THE STRESS SPACES OF BIPARTITE FRAMEWORKS

By

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For my beautiful, loving, amazingly patient wife, Vicky and my wonderful son, Zachary Alan.

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS iii

ABSTRACT v

CHAPTERS

1 INTRODUCTION 1

2 BACKGROUND 3
2.1 Projective d-Space 3
2.2 Statics 4
2.3 Bar and Joint Frameworks 6
2.4 Bipartite Bar and Joint Frameworks 9

3 MAIN RESULTS 15
3.1 Preliminaries 15
3.2 Frameworks With Complete Bipartite Spanning Subframeworks . 19
3.3 Other Realizations 25

4 GENERAL RESULTS 39
4.1 The Kernel 39
4.2 The Image 50

5 EXAMPLES 52
5.1 K_{44} Minus One Bar 52
5.2 K_{44} Minus Two Bars 60

6 CONCLUSION 71
6.1 Summary 71
6.2 Future Research 72

REFERENCES 74

BIOGRAPHICAL SKETCH 75

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In this dissertation we extend the work done by Bolker and Roth in calculating the dimension of the stress space of bipartite frameworks.

We have presented results which are analogous to those known for complete bipartite frameworks, yet hold for a much wider class of bipartite frameworks.

The main results give the dimension of the stress space of certain classes of bipartite frameworks, which are easily calculated using only the number of bars and joints and knowledge of the geometry of the specific realization.

Some of the techniques used in this dissertation are similar to the techniques used by Bolker and Roth. In particular we use the same map τ to separate each problem into two parts.

We also have made some observations on dependencies of sets of tensor products of points from a vector space. These results, although interesting for their own sake, are used to establish some general results about the kernel of the map τ.

CHAPTER 1
INTRODUCTION

Most people have an intuitive idea of what it means when we say that a structure is rigid. Anyone who has ever built a bookcase, swing set, or scaffold has faced the question of the rigidity of the structure at hand. For example, in the case of a scaffold one may want to build a framework which looks like the edge skeleton of a cube as seen in figure 1.1.

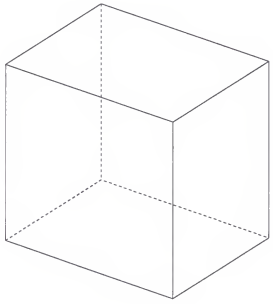


Figure 1.1: Edge Skeleton of the Cube

When building such a structure we know that without extra bracing this framework will likely collapse. We would like to know how to brace this scaffold so that it will be rigid. One might be surprised to discover that adding diagonals to five of the faces of the cube as seen in figure 1.2 leaves a framework which is still not rigid. In fact, from rigidity theory, we know that for a framework in 3-space with eight joints

we would need at least 18 bars [3, 5] to build a framework which is rigid. One can easily see that the framework in Figure 1.2 has only 17 bars. Hence the scaffold seen below is not rigid.

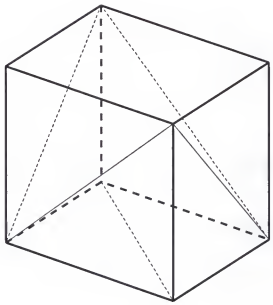


Figure 1.2: Braced Cube

In combinatorial rigidity one would like to answer questions of rigidity in a purely combinatorial way. This involves determining whether or not a given framework is rigid by simply counting the number of bars and joints. However, as one might guess, this is not always possible. There are often special positions of the joints of a framework for which the entire framework becomes flexible. Hence, in studying rigidity, we often must take into account geometric information as well.

The purpose of this dissertation is to examine the special class of frameworks called bipartite frameworks and determine when such frameworks are rigid.

CHAPTER 2
BACKGROUND

This chapter is devoted to the background and notation needed to understand the work presented in this dissertation. Some of the notation introduced in this chapter is not used for several pages.

2.1 Projective d-Space

We define the points of real projective d-space, $\mathbb{P}^d$, to be the nonzero vectors from $\mathbb{R}^{d+1}$ modulo the following equivalence relation: if $a, b \in \mathbb{R}^{d+1}$ then a is equivalent to b if and only if $a = \mu b$ with $\mu \in \mathbb{R}$ and $\mu \neq 0$.

Next, we define the standard representative of an element $a \in \mathbb{P}^d$. Let a_k be the last nonzero coordinate position of a. Define the standard representative of a to be $a' = \frac{1}{a_k} a$. If $k = d + 1$, we call a a finite point. If, on the other hand, $k < d + 1$ then $a_{d+1} = 0$, and we call a a point at infinity. The standard embedding of $\mathbb{R}^d$ into $\mathbb{P}^d$ is given by

$$(a_1, a_2, a_3, ..., a_d) \mapsto (a_1, a_2, a_3, ..., a_d, 1).$$

We say a set of points $W = \{w_1, w_2, w_3, ..., w_n\} \subset \mathbb{P}^d$ is dependent if the standard representatives are linearly dependent.

We now define the Plücker coordinates of the line $\mathcal{L}$ through 2 distinct points $a, b \in \mathbb{P}^d$. First, let M be the $2 \times (d + 1)$ matrix whose two rows are the projective coordinates of a and b respectively. The Plücker coordinates for $\mathcal{L}$ are denoted by $L = a \vee b$ and are given by the $\binom{d+1}{2}$-tuple whose entries are the 2×2 minors of M in some fixed order.

One can show that if $c \in \mathbb{P}^d$ is on the line through a and b then $a \vee b = \nu a \vee c$ for some nonzero $\nu \in \mathbb{R}$ [13]. Furthermore, it can be shown that distinct lines have Plücker coordinates which are not scalar multiples of one and other [9].

Now, given j points, $x_1, x_2, ..., x_j$, in $\mathbb{P}^d$, let N be the $j \times (d+1)$ matrix whose j rows are the projective coordinates of $x_1, x_2, ..., x_j$. Define the Plücker coordinates of $x_1 \vee x_2 \vee ... \vee x_j$ to be the $\binom{d+1}{j}$-tuple whose entries are the $j \times j$ minors of N. In the case when $j = d+1$ we use the following notation:

$$x_1 \vee x_2 \vee x_3 \vee ... \vee x_{d+1} = [x_1 x_2 x_3 ... x_{d+1}].$$

We note that $\vee$ is antisymmetric; hence $\vee_{i=1}^{j} x_i = 0$ if and only if the set $\{x_1, x_2, x_3, ...x_j\}$ is dependent. Furthermore, for any $\nu \in \mathbb{R}$ we have $a \vee (\nu b + c) = \nu a \vee b + a \vee c$. We often will use juxtaposition to indicate $\vee$; that is $a \vee b = ab$.

Using projective coordinates we can easily keep track of forces and moments. Moreover, the properties we are studying are invariant under projective transformations. For the remainder of the dissertation we will work with the finite points of $\mathbb{P}^d$. For an in-depth study of projective geometry see Veblen and Young [13] or Hartshorne [8]. Also, one can can find more information on the properties of $\vee$ and $\otimes$ in Doubilet et al. [6], Greub [7], or Meyer et al. [9].

2.2 Statics

In statics a set of forces, often represented as vectors, is applied to a structure at specified points. The response of the structure to these forces is then studied.

If we apply a set of forces to a solid object, in 3-space, the motion of the object will be determined by the net moment of the applied forces. In 3-space a nonzero moment about a particular axis indicates a rotation about that axis. We eliminate all Euclidean motions, in 3-space, by restricting our attention to sets of forces which cause no motion when applied to a solid object. In particular, we need more than just the sum of the forces, as free vectors, to be zero. We see in figure 2.1 that the sum

of the forces is zero; yet the bar to which the forces are applied obviously will move. In this example we notice that the forces are bound to a particular line of action and it is these line-bound forces that we will work with when examining motions. In particular, to cause no motions, a set of forces applied to a solid object along specified lines must have a zero net moment about every axis. We call such a set of forces an *equilibrium system.*

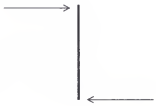


Figure 2.1: Line-Bound Forces

Moreover, let $G = \{\mathbf{g}_i\}$ be a set of forces in 3-space applied at the Euclidean points $\mathbf{q}_i$. Then G is an equilibrium system if and only if the following equations are satisfied:

1. $\sum_i \mathbf{g}_i = 0$
2. $\sum_i \mathbf{g}_i \times \mathbf{q}_i = 0$

Now, we translate this to a projective setting. A free vector $\mathbf{f} = (f_1, f_2, f_3)$ applied at a point $\mathbf{p} = (p_1, p_2, p_3)$ can be represented projectively. Let $\bar{f} = (f_1, f_2, f_3, 0)$ and $\bar{p} = (p_1, p_2, p_2, 1)$. Look at $F = \bar{f} \vee \bar{p} = (f_1, f_2, f_3, f_1p_2 - f_2p_1, f_1p_3 - f_3p_1, f_2p_3 - f_3p_2)$ and note that the first three coordinates of F are the same as those of $\mathbf{f}$ and that the last three coordinates are simply the coordinates of $\mathbf{f} \times \mathbf{p}$. We call F a line-bound force or simply a force. We see that the coordinates of F have all the essential information to determine both the net force and the net moment. Therefore a system of forces F_i is an equilibrium system if and only if $\sum_i F_i = 0$. This projective version of line-bound forces generalizes nicely to higher dimensions where the cross product is meaningless.

2.3 Bar and Joint Frameworks

Instead of applying forces to solid objects we will apply forces to objects called bar and joint frameworks. A bar and joint framework is a pair $\mathcal{S}(J, E)$, where J is an indexed set of points $\{a_1, a_2, a_3, ..., a_v\}$ from $\mathbb{P}^d$, called the joints, and E is a set of unordered pairs $\{\{a_i, a_j\}, ...\}$ called the bars. This also defines an obvious graph associated with the bar and joint framework called the underlying graph. For simplicity, J will contain only distinct finite points of $\mathbb{P}^d$ and $i \neq j$ for every $\{a_i, a_j\} \in E$. We are interested in static rigidity, which is the study of the response of bar and joint frameworks to all equilibrium systems. Since an equilibrium system will cause no Euclidean motions we will try to determine when a framework has an internal motion.

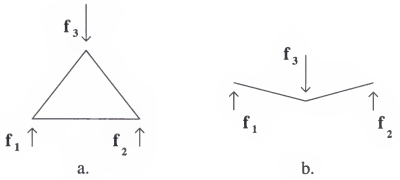


Figure 2.2: Motions

To illustrate, we look at the two frameworks seen in figure 2.2 with forces $\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{j}$ and $\mathbf{f}_3 = -2\mathbf{j}$ applied as depicted. Clearly, the set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an equilibrium system. It also is obvious that figure 2.2a will not move, while the two bars in figure 2.2b will fold together. This is an example of an internal motion.

We first note that if forces are applied to a bar and joint framework we can always represent this as a system of forces composed of exactly one force at each joint.

DEFINITION 2.3.1. *An* equilibrium load *on a bar and joint framework is an assignment of forces F_i to the joints a_i such that F_i passes through a_i and $\sum_i F_i = 0$.*

Notice that the set of equilibrium loads forms a vector space over $\mathbb{R}$.

DEFINITION 2.3.2. *A* resolution *of the equilibrium load F_i by the bar and joint framework is an assignment of scalars $\lambda_{a_i a_j}$ such that for each $a_i \in J$ we have*

$$F_i + \sum_j \lambda_{a_i a_j} a_i \vee a_j = 0,$$

where $\lambda_{a_i a_j} = 0$ if $\{a_i, a_j\} \notin E$.

Thus a resolution can be thought of as an assignment of tensions and compressions to the bars of a bar and joint framework, which are in equilibrium with the equilibrium load at each joint.

Figure 2.3: Bar and Joint Framework

DEFINITION 2.3.3. *The* rigidity matrix, *$R(\mathcal{S})$, of a bar and joint framework $\mathcal{S}$ is the matrix whose columns are indexed by the joints of $\mathcal{S}$ and whose rows are indexed by the bars of $\mathcal{S}$ with entries*

$$R_{e,a_i} = \begin{cases} a_i \vee a_j & \text{if } e = \{a_i, a_j\} \in E, \\ 0 & \text{if } a_i \notin e. \end{cases}$$

Note that each $a_i \in J$ represents $\binom{d+1}{2}$ columns. For instance, the framework seen in figure 2.3 has the following rigidity matrix:

$$R = \begin{array}{c} \\ \{a,b\} \\ \{a,c\} \\ \{b,c\} \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} a \vee b & b \vee a & 0 \\ a \vee c & 0 & c \vee a \\ 0 & b \vee c & c \vee b \end{pmatrix} \end{array}.$$

Each possible assignment of scalars to the bars matches some equilibrium load $F = \{F_1, F_2, F_3, ..., F_v\}$. To see this, let λ be an assignment of scalars to the bars of

some framework $\mathcal{S}$. Define $-F_i = \sum_j \lambda_{a_i a_j} a_i \vee a_j$ The following shows that F is in fact an equilibrium load:

$$\begin{aligned}\sum_i F_i &= \sum_i \sum_j \lambda_{a_i a_j} a_i \vee a_j \\ &= \sum_{(i,j),\, i<j} (\lambda_{a_i a_j} a_i \vee a_j + \lambda_{a_j a_i} a_j \vee a_i) \\ &= 0.\end{aligned}$$

From this we conclude that the row space of R is contained in the space of equilibrium loads. Now we would like to answer the following question: Are all equilibrium loads resolved?

DEFINITION 2.3.4. *A bar and joint framework is* statically rigid *if every equilibrium load is resolved.*

Therefore to be statically rigid the row space of $R(\mathcal{S})$ must contain a basis for the space of equilibrium loads.

DEFINITION 2.3.5. *A* stress *of a bar and joint framework is an assignment of scalars $\lambda_{a_i a_j}$ to the bars, at least one nonzero, such that at each point a_i we have*

$$\sum_j \lambda_{a_i a_j} a_i \vee a_j = 0$$

where, again, we have $\lambda_{a_i a_j} = 0$ if $\{a_i, a_j\} \notin E$.

Note, the scalars of a stress are the coefficients of a row dependence of $R(\mathcal{S})$. A stress also can be thought of as a nontrivial set of tensions and compressions on each bar of the framework which reaches equilibrium with the zero load.

It is clear that the stresses, together with the all zero assignment, form a vector space over the reals. We call this vector space the stress space. Given a bar and joint framework G, we denote the stress space of G by Ω_G. Hence, to determine if a framework is statically rigid, one compares the rank of the rigidity matrix to the dimension of the space of equilibrium loads. We find for a given framework G $rank(R) = |E| - dim(\Omega_G)$. From this we deduce the obvious corollary.

COROLLARY 2.3.6. *Let G be a bar and joint framework. Let G' be a framework obtained by removing one bar from G. Then $dim(\Omega_G) \geq dim(\Omega_{G'}) \geq dim(\Omega_G) - 1$.*

What is now left to do is calculate the dimension of the space of equilibrium loads. We note that if the framework is in projective d-space then there is a d-dimensional choice of possible forces at each joint. Since the forces in an equilibrium load must sum to zero and each force in the load is a $\binom{d+1}{2}$-tuple, each load is subject to $\binom{d+1}{2}$ restrictions. Thus, if $|J| = v$ then the dimension of the space of equilibrium loads is $dv - \binom{d+1}{2}$; as long as the framework does not lie in a $d-2$ dimensional subspace. This brings us to an important theorem [5].

THEOREM 2.3.7. *S is statically rigid if and only if the rank of the rigidity matrix is $dv - \binom{d+1}{2}$.*

Now, using Theorem 2.3.7 we arrive at the following corollary.

COROLLARY 2.3.8. *If S is statically rigid then $|E| \geq dv - \binom{d+1}{2}$.*

Details of the basics of rigidity theory can be found in both Crapo [3] and Crapo and Whiteley [5].

2.4 Bipartite Bar and Joint Frameworks

We will now concentrate on a special type of framework called a bipartite framework.

DEFINITION 2.4.1. *A* bipartite framework *is a framework whose underlying graph is a bipartite graph with vertex sets A and B. Unless otherwise stated, we will always assume $|A| = m$ and $|B| = n$.*

For the remainder of this paper we use K_{mn} to denote both the complete bipartite framework and the underlying complete bipartite graph K_{mn}.

A graph is bipartite if and only if it has only odd cycles [2]. The rigidity of bipartite frameworks has been studied for years. One reason bipartite frameworks are of such interest is because they are examples of frameworks that contain no triangles.

In fact, around the turn of the century it was known that the framework K_{33} in the plane was stress-free if the joints were in generic position; yet if the 6 joints of the framework fell on a conic, there would be a stress and the framework would become nonrigid or flexible. Moreover, in 1978 Whiteley made a similar conjecture for K_{46} in 3-space. In 1980, Bolker and Roth wrote the paper "When is a bipartite graph a rigid framework?" [1] which gives a formula for the dimension of the stress space of any realization of a complete bipartite framework in d-space. The result of Bolker and Roth will be stated after some notation is presented.

We often will consider different realizations of a particular framework. A realization is an assignment of joints to points in $\mathbb{P}^d$, the space in which the framework is embedded.

For a bipartite framework G one may write a stress as a real valued function λ_{ab} on the bars such that

1. $\sum_{a\in A} \lambda_{ab} ab = 0$ for all $b \in B$

2. $\sum_{b\in B} \lambda_{ab} ab = 0$ for all $a \in A$

where $\lambda_{ab} = 0$ if $\{a, b\} \notin G$. It is easy to check that this definition of a stress, in the bipartite case, coincides with the definition given earlier; that is, every matrix λ satisfying conditions 1 and 2, given above, is clearly a row dependence of the rigidity matrix of the framework G.

Therefore, a stress λ of a bipartite framework G with vertex sets A and B is an $|A| \times |B|$ matrix whose rows are indexed by the elements of A and columns are indexed by the elements of B. Hence, the stress space Ω_G is the space of $|A| \times |B|$ matrices which satisfies equations 1 and 2, given above,with zeros in the prescribed positions. Note that every bipartite framework G can be described as some complete bipartite framework K_{mn} with specified bars removed.

Let G be a bipartite framework and $\lambda \in \Omega_G$. Denote $\rho_a = \sum_{b\in B} \lambda_{ab}$ for every $a \in A$ and $\gamma_b = \sum_{a\in A} \lambda_{ab}$ for every $b \in B$, respectively called the row sums and

column sums of a stress. Let $A_b(G)$ and $B_a(G)$ be defined as follows:

$$A_b(G) = \{a \in A | \{ab\} \in G\}$$

$$B_a(G) = \{b \in B | \{ab\} \in G\}.$$

Given a bipartite framework on the vertex sets $A = \{a_1, a_2, a_3, ..., a_m\}$ and $B = \{b_1, b_2, b_3, b..., b_n\}$ we can define the linear map $\tau : \Omega_G \to \mathbb{R}^{m+n}$ by

$$\tau(\lambda) = (\rho_{a_1}, \rho_{a_2}, \rho_{a_3}, ..., \rho_{a_m}, \gamma_{b_1}, \gamma_{b_2}, \gamma_{b_3}, ..., \gamma_{b_n}).$$

The space of dependencies will be important in our analysis of the stress space of bipartite frameworks. Let $S = \{s_1, s_2, s_3, ..., s_t\} \subset \mathbb{P}^d$; fix the homogeneous coordinates of the points of S; and define

$$\mathbb{D}(S) = \{(\alpha_1, \alpha_2, \alpha_3, ...\alpha_t) | \sum_{i=1}^{t} \alpha_i s_i = 0\} \subset \mathbb{R}^t.$$

Note that the t-tuple of zeros is in $\mathbb{D}(S)$. Furthermore, $\mathbb{D}(S)$ is closed under addition and scalar multiplication and therefore a vector space over $\mathbb{R}$. Hence we call $\mathbb{D}(S)$ the vector space of dependencies of S.

We also will make use of the Kronecker product of an l-tuple with a k-tuple which is defined as follows: given

$$a = (a_1, a_2, a_3, ...a_l) \in \mathbb{P}^{l-1}$$

and

$$b = (b_1, b_2, b_3, ..., b_k) \in \mathbb{P}^{k-1}$$

define $a \otimes b$ to be the $l \times k$ matrix M with $M_{ij} = a_i \cdot b_j$. Moreover, given two sets $E \subset \mathbb{P}_{l-1}$ and $F \subset \mathbb{P}_{k-1}$ we define $E \otimes F$ to be the vector space of finite linear combinations of elements of the form $e \otimes f$ with $e \in E$ and $f \in F$.

In analyzing the stress space of a bipartite framework we will use the notation $< A >$ to mean the subspace of $\mathbb{P}^d$ spanned by the points of $A \subset \mathbb{P}^d$ and make use of the following set

$$C_G = \{a \in A | a \in < B_a(G) >\} \cup \{b \in B | b \in < A_b(G) >\}.$$

We will use the following notations:

$$\mathbb{D}(C_G \otimes C_G) = \mathbb{D}(\{c \otimes c | c \in C_G\}).$$

Furthermore, a set $S = \{s_1, s_2, ..., s_t\}$ which has the property that the set $\{s_1 \otimes s_1, s_2 \otimes s_2, ..., s_t \otimes s_t\}$ is dependent is said to be conically dependent. If we are given a set A of n points from $\mathbb{P}^d$ we will denote by $N(A)$ the dimension of the null space or *nullity* of the $n \times (d+1)$ matrix whose rows are the projective coordinates of the points of A. Notice that with this notation we have $dim(\mathbb{D}(A)) = N(A)$. We call t_1 an *isthmus* of the set $T = \{t_1, t_2, t_3, ..., t_k\} \subset \mathbb{P}^d$ if t_1 is not involved in any dependence on the set T.

Let K be a complete bipartite framework on the vertex sets A and B such that $|A| = m$ and $|B| = n$. Define $C = (A \cap < B >) \cup (< A > \cap B)$. Note that $C = C_{K_{mn}}$. Now we may state the result of Bolker and Roth.

THEOREM 2.4.2. $dim(\Omega_K) = N(A)N(B) + dim(\mathbb{D}(C \otimes C))$.

The main goal of this dissertation is to generalize the above result. A large part of our examination of the stress space of general bipartite frameworks will require us to determine $dim(\mathbb{D}(C_G \otimes C_G))$. It can be shown that $\mathbb{D}(C_G \otimes C_G)$ is closely related to the quadric polynomials which vanish on the set C_G.

DEFINITION 2.4.3. *A* quadric polynomial *is a polynomial of the form*

$$p(x_1, x_2, ..., x_m) = \sum_{1 \le i \le j \le m} \eta_{ij} x_i x_j + \sum_{1 \le i \le m} \eta_i x_i + \eta.$$

We note that this quadric polynomial is completely determined by its $\binom{m+2}{2}$ coefficients.

DEFINITION 2.4.4. *A* quadric surface *is the set of zeros of a nontrivial quadric polynomial.*

We denote by $Q(C)$ the space of quadric polynomials which vanish on the set C.

The following may be proved:

LEMMA 2.4.5. *If* $C = \{c_1, c_2, c_3, ..., c_k\} \subset \mathbb{P}^d$ *and* $h = dim(< C >) \leq d$ *then* $dim(Q(C)) \geq max\{0, \binom{h+2}{2} - k\}$.

THEOREM 2.4.6. *If* $C = \{c_1, c_2, c_2, ...c_k\} \subset \mathbb{P}^d$ *and* $h = dim(< C >) \leq d$ *then* $dim(\mathbb{D}(C \otimes C)) = dim(Q(C)) + k - \binom{h+2}{2}\}$.

The details of these results can be found in Bolker and Roth [1] and Semple and Roth [10].

In calculating $dim(\mathbb{D}(C \otimes C))$ we use the following map

$$V : \mathbb{P}^d \to \mathbb{P}^{\binom{d+2}{2}-1}$$

called the Veronese map. If $(x_1, x_2, x_3, ..., x_{d+1}) \in \mathbb{P}^d$ then $V((x_1, x_2, x_3, ..., x_{d+1})) = (x_1^2, x_2^2, x_3^2, ..., x_{d+1}^2, x_1x_2, x_1x_3, ..., x_dx_{d+1})$. We would like to answer the following question: When is the image of a set of points under V dependent in $\mathbb{P}^{\binom{d+2}{2}-1}$? For instance, we know if a set C of 6 points is in generic position in $\mathbb{P}^2$ then $dim(\mathbb{D}(C \otimes C)) = 0$. If the six points of C all lie on a conic, then $dim(\mathbb{D}(C \otimes C)) = 1$. In fact, if $d = 2$ we know the following:

1. The image of 2 points spans a one-dimensional subspace of $\mathbb{P}^5$.

2. The image of 3 points spans a two-dimensional subspace of $\mathbb{P}^5$.

3. The image of any number of points on a line spans a two-dimensional subspace of $\mathbb{P}^5$.

4. The image of 4 points, which are not collinear, spans a three-dimensional subspace of $\mathbb{P}^5$.

5. The image of 5 points, no 4 collinear, spans a four-dimensional subspace of $\mathbb{P}^5$.

6. The image of 6 points, not all on a conic, spans $\mathbb{P}^5$.

Furthermore, given any set of points $C \subset \mathbb{P}^2$ we can examine the above list and determine the dimension of $\mathbb{D}(C \otimes C)$ [4].

CHAPTER 3
MAIN RESULTS

3.1 Preliminaries

For the remainder of this paper we will assume that the vertex sets involved are A and B. Moreover, Let $A_1 \subset A$ and $B_1 \subset B$ be a basis for $< A >$ and $< B >$ respectively. We will also denote by $B_2 = B - B_1$. We note here that we will use λ to mean several different things. It will be clear from the context which we are using. We will think of the stress λ as an $m \times n$ matrix whose rows are indexed by the elements of A and whose columns are indexed by the elements of B, as well as a real-valued function $\lambda : A \times B \to \mathbb{R}$. We will freely use λ_{ab} to mean $\lambda(a, b)$ when λ is a function and to mean the (a, b) entry of λ when it is being thought of as a matrix.

The following lemma is proved in the paper by Bolker and Roth [1]. It is independent of the framework G. The proof is presented here because it is crucial to the proofs of many of the subsequent theorems.

LEMMA 3.1.1. *Assume that λ is a matrix $\lambda : A \times B \to \mathbb{R}$, and that ρ and γ are maps $\rho : A \to \mathbb{R}$ and $\gamma : B \to \mathbb{R}$. If*

(1) $\sum_a \lambda_{ab} a = \gamma_b b$ *for all* $b \in B_2$,

and

(2) $\sum_b \lambda_{ab} b = \rho_a a$ *for all* $a \in A$,

then the following are equivalent:

(i) $\sum_a \lambda_{ab} a = \gamma_b b$ *for all* $b \in B_1$,

(ii) $\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$.

Proof. Assuming (i), together with (1) we obtain

$$(iii) \quad \sum_{a\in A} \lambda_{ab} a = \gamma_b b \qquad \text{for all } b \in B.$$

We begin with the trivial statement,

$$\sum_{a\in A}\sum_{b\in B} \lambda_{ab} a \otimes b = \sum_{b\in B}\sum_{a\in A} \lambda_{ab} a \otimes b.$$

Linearity of the tensor product gives us

$$\sum_{a\in A} a \otimes (\sum_{b\in B} \lambda_{ab} b) = \sum_{b\in B} (\sum_{a\in A} \lambda_{ab} a) \otimes b$$

and using (iii) and (2) we conclude

$$\sum_{a\in A} \rho_a a \otimes a = \sum_{b\in B} \gamma_b b \otimes b.$$

Conversely, rewrite (ii) as

$$\sum_{a\in A} \rho_a a \otimes a - \sum_{b\in B_2} \gamma_b b \otimes b = \sum_{b\in B_1} \gamma_b b \otimes b.$$

Using (1) and (2) we find

$$\sum_{a\in A}\sum_{b\in B} \lambda_{ab} a \otimes b - \sum_{a\in A}\sum_{b\in B_2} \lambda_{ab} a \otimes b = \sum_{b\in B_1} \gamma_b b \otimes b.$$

Again we use linearity of the tensor product to obtain

$$\sum_{a\in A} (\sum_{b\in B} \lambda_{ab} a \otimes b - \sum_{b\in B_2} \lambda_{ab} a \otimes b) = \sum_{b\in B_2} \gamma_b b \otimes b.$$

The above can be rewritten as:

$$\sum_{a\in A}\sum_{b\in B_1} \lambda_{ab} a \otimes b - \sum_{b\in B_1} \gamma_b b \otimes b = 0.$$

Finally we arrive at

$$\sum_{b\in B_1} (\sum_{a\in A} \lambda_{ab} a - \gamma_b b) \otimes b = 0.$$

Because B_1 is independent, we may conclude [7,p. 6]

$$\sum_{a\in A} \lambda_{ab} a = \gamma_b b \qquad \text{for all } b \in B_1,$$

which is the desired result. ∎

LEMMA 3.1.2. *If G is a bipartite framework, then $ker(\tau)_G \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$*

Proof. Let $\lambda \in ker(\tau)$. Note that every row of λ is an element of $\mathbb{D}(B)$ and every column of λ is an element of $\mathbb{D}(A)$. We will prove $\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ by induction on the ordering of the indices stated below.

Order the entries M_{ij} of an $m \times n$ matrix M, according to the following ordering on the indices: M_{ij} is earlier than M_{kl} if

1. $i < k$ or

2. $i = k$ and $j < l$.

Let λ_{ij} be the earliest nonzero entry of λ and construct the following element of $\mathbb{D}(A) \otimes \mathbb{D}(B)$: Let $\mu_B = (\lambda_{i1}, \lambda_{i2}, ..., \lambda_{in}) \in \mathbb{D}(B)$ and let $\mu_A = (\lambda_{1j}, \lambda_{2j}, ..., \lambda_{mj}) \in \mathbb{D}(A)$. Hence, $w = \frac{1}{\lambda_{ij}} \mu_A \otimes \mu_B \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ and therefore $N = \lambda - w$ is a matrix whose rows and columns are still elements of $\mathbb{D}(A)$ and $\mathbb{D}(B)$ respectively. Now we show $N_{kl} = 0$ if N_{kl} is earlier in the ordering than N_{ij}. Assume first $N_{kl} \neq 0$ with $N_{kl} < N_{ij}$. This implies that

$$w_{kl} = \frac{\lambda_{il}\lambda_{kj}}{\lambda_{ij}} \neq 0.$$

Therefore both $\lambda_{il} \neq 0$ and $\lambda_{kj} \neq 0$. Now, if $k < i$, then $\lambda_{kj} \neq 0$, which contradicts the fact that λ_{ij} was the earliest nonzero entry. On the other hand, if $k = i$ and $l < j$ then $\lambda_{il} \neq 0$ brings us to the same contradiction. Therefore, the earliest nonzero position of N is later than N_{ij}. Hence, by induction we can write λ as a linear combination of elements from $\mathbb{D}(A) \otimes \mathbb{D}(B)$. Consequently $ker(\tau) \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$ as required. ■

We note that this gives us an upper bound on $dim(ker(\tau)_G)$. The following theorem will similarly give an upper bound for $dim(Im(\tau)_G)$.

LEMMA 3.1.3. *Let G be a bipartite framework. Then $Im(\tau)_G \subset \mathbb{D}(C_G \otimes C_G)$.*

Proof. Let $\lambda \in \Omega_G$. Hence, we have

1. $\sum_a \lambda_{ab} a = \gamma_b b$,

2. $\sum_b \lambda_{ab} b = \rho_a a$

where $\lambda_{ab} = 0$ if $\{a, b\} \notin G$. Hence $\tau(\lambda) = (\rho_a, ..., \gamma_b, ...)$. Now, λ satisfies conditions 1, 2, and i of Lemma 3.1.1. Therefore we may conclude

$$\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$$

and

$$Im(\tau)_G \subset \mathbb{D}((A \cup B) \otimes (A \cup B)).$$

We note that from 1 we conclude if $b \notin C_G$ then $\gamma_b = 0$. Similarly, 2 yields, if $a \notin C_G$ then $\rho_a = 0$. Hence $\tau(\lambda) \in \mathbb{D}(C_G \otimes C_G)$. Therefore $Im(\tau)_G \subset \mathbb{D}(C_G \otimes C_G)$. ■

In general, we would like to determine for which frameworks the reverse inclusion holds, that is, to find geometric conditions for when $\mathbb{D}(C_G \otimes C_G) \subset Im(\tau)_G$. In fact, many of the following theorems are proved using similar techniques.

The following is a sketch of one of the most common proof techniques. This is presented for the sake of clarity and it allows us to slightly shorten the explanation of parts of several of the following proofs.

To show $\mathbb{D}(C_G \otimes C_G) \subset Im(\tau)_G$ for a bipartite framework G, we start with $w = (\rho_a, ..., \gamma_b, ...) \in \mathbb{D}(C_G \otimes C_G)$ and construct $\lambda \in \Omega_G$ such that $\tau(\lambda) = w$. To do this, we start by defining the elements of the columns of λ corresponding to each element $b \in B_2$ so that they satisfy the requirements of a stress on G. That is, for each $b \in B_2$ define λ_{ab} so that

$$\sum_a \lambda_{ab} a = \gamma_b b$$

and $\lambda_{ab} = 0$ if $\{a, b\} \notin G$. Note we are always able to to this because we have $\gamma_b b \in < A_b(G) >$. Similarly, for each $a \in A$, if possible as seen in figure 3.1, we define the remaining elements of each row so that each row satisfies the requirements of a stress on G. Once we have done this we have constructed a matrix which satisfies conditions 1, 2, and ii of Lemma 3.1.1. Hence we may conclude that the matrix we defined is in fact a stress of the framework G.

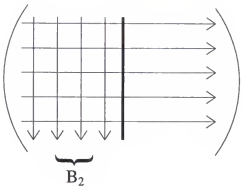


Figure 3.1: λ Construction

3.2 Frameworks With Complete Bipartite Spanning Subframeworks

In the first few theorems we will be interested in realizations of bipartite frameworks on the vertex set A and B which have the following property:

DEFINITION 3.2.1. *We say a bipartite framework G has a* complete bipartite spanning subframework *if there exists a complete bipartite subframework of G on the vertex sets A and $B_1 \subset B$, such that B_1 is a basis for $< B >$.*

We will solve many of these problems in two parts. Using the linear map τ, defined previously, we know that $dim(\Omega_G)$ is given by $dim(ker(\tau)_G) + dim(Im(\tau)_G)$.

THEOREM 3.2.2. *Let G be a bipartite graph with a complete bipartite spanning subframework. Order the elements of the set B so that $B_1 = \{b_{p+1}, b_{p+2}, b_{p+3}, ..., b_n\}$. Then $dim(ker(\tau))_G = \sum_{j=1}^{p} dim(\mathbb{D}(A_{b_j}(G)))$.*

Proof. Take the following elements as a basis for $\mathbb{D}(B)$:

$$w_j = (0, 0, ..., 1, ..., 0, \beta_{p+1}, \beta_{p+2}, ..., \beta_n)$$

where the 1 is in the j^{th} position. For each $j = 1, 2, 3, ..., p$ choose $\{f_{1j}, f_{2j}, ...f_{k_j j}\}$, a basis for $\mathbb{D}(A_{b_j}(G))$. We know

1. $k_j = dim(\mathbb{D}(A_{b_j}(G)))$ and
2. $f_{ij} \in \mathbb{D}(A)$.

Let $\mathcal{D} = \{D_{ij}\}$ be the set of elements from $\mathbb{D}(A) \otimes \mathbb{D}(B)$ given by

$$D_{ij} = f_{ij} \otimes w_j.$$

It will be convenient to define the following: for any matrix M let $C_k(M)$ denote the k^{th} column of M. Note, by construction, $C_h(D_{ij}) = \mathbf{0}$ for $h \leq p$ except when $h = j$, in which case $C_j(D_{ij}) = f_{ij}$.

We see $|\mathcal{D}| = \sum_{j=1}^{p} dim(\mathbb{D}(A_{b_j}(G)))$. To see that $\mathcal{D}$ is independent, first assume

$$\sum_j \sum_i \nu_{ij} D_{ij} = 0.$$

Since $C_j(\sum_j \sum_i \nu_{ij} D_{ij}) = \sum_i \nu_{ij} f_{ij}$, we can conclude for each $j = 1, 2, 3, ..., p$ that we have $\sum_i \nu_{ij} f_{ij} = 0$. Therefore $\nu_{ij} = 0$ for every i and j because, for each j, $\{f_{1j}, f_{2j}, f_{3j}, ..., f_{kj}\}$ is independent. Hence, $\mathcal{D}$ itself is independent.

Finally, we show $< \mathcal{D} >= ker(\tau)_G$. Again, by construction, every element of $\mathcal{D}$ has zeros in the positions i, j where $\{a_i, b_j\} \notin G$, hence for any coefficients ν_{ij} we have

$$\sum_{ij} \nu_{ij} D_{ij} \in ker(\tau)_G.$$

Therefore, $< \mathcal{D} > \subseteq ker(\tau)_G$.

Conversely, let $\lambda \in ker(\tau)_G$. We will show that there are coefficients ν_{ij} such that $\lambda = \sum_{ij} \nu_{ij} D_{ij}$. For a fixed $1 \leq j \leq p$ we have $C_j(\lambda) \in \mathbb{D}(A_{b_j}(G))$. Therefore, there are coefficients ν_{ij} such that $C_j(\lambda) = \sum_{i=1}^{k_j} \nu_{ij} C_j(D_{ij})$. Hence, $C_j(\lambda - \sum_{i=1}^{k_j} \nu_{ij} D_{ij}) = \mathbf{0}$. Therefore, $C_j(\lambda - \sum_j \sum_{i=1}^{k_j} \nu_{ij} D_{ij}) = \mathbf{0}$ for $j = 1, 2, 3, ..., p$.

Let $E = \lambda - \sum_j \sum_{i=1}^{k_j} \nu_{ij} D_{ij}$. We see E is a linear combination of $m \times n$ matrices which have the property that each of their rows are elements of $\mathbb{D}(B)$. Therefore E itself has this property. Now $C_j(E) = \mathbf{0}$ for $j = 1, 2, 3, ..., p$ and each row of E is an element of $\mathbb{D}(B)$, which can only be nonzero on a basis of $< B >$. Hence $E = \mathbf{0}$.

We have shown $ker(\tau)_G =< \mathcal{D} >$. Thus, $dim(ker(\tau)_G) = \sum_j dim(\mathbb{D}(A_{b_j}(G)))$.

■

Theorem 3.2.2 gives the $dim(ker(\tau)_G)$ for any realization of a framework containing a complete bipartite spanning subframework. Next we find the $dim(Im(\tau)_G)$ for these frameworks.

THEOREM 3.2.3. *A pair of vectors* $(\rho_a, a \in A)$ *and* $(\gamma_b, b \in B)$ *are row and column sums of a stress of a bipartite framework* G *with a complete spanning subframework if and only if*

(i) $\sum_a \rho_a a \otimes a = \sum_b \gamma_b b \otimes b$

and

(ii) $\rho_a = 0$ *if* $a \notin < B_a(G) >$, $\gamma_b = 0$ *if* $b \notin < A_b(G) >$.

Proof. Let $\lambda \in \Omega_G$. Then its row and column sums ρ_a and γ_b satisfy

(1) $\sum_a \lambda_{ab} a = \gamma_b b$ and

(2) $\sum_b \lambda_{ab} b = \rho_a a$.

Using Lemma 3.1.1 we see that property (i) holds for a stress of any bipartite framework.

Furthermore, since λ is a stress, it is clear from (2) that if $a \notin < B_a(G) >$ then $\rho_a = 0$. Similarly, using 1 we find $\gamma_b = 0$ whenever $b \notin < A_b(G) >$.

Conversely, assume i and ii, and let $B_1 \subset B$ be the basis of $< B >$ so that the subframework on the sets A and B_1 is complete.

Note, for every $s \in B_2 = B - B_1$, we know $\gamma_s s \in < A_s(G) >$. Hence, for each $s \in B_2$, there exist scalars λ_{as} such that

$$\sum_a \lambda_{as} a = \gamma_s s$$

where $\lambda_{as} = 0$ if $\{a, s\} \notin G$.

Furthermore, since $\rho_a a \in < B_a > \subset < B >$ we have

$$\rho_a a - \sum_{s \in B_2} \lambda_{as} s \in < B > \qquad \text{for all } a \in A.$$

Hence, for every $a \in A$, there exist scalars λ_{ax} with $x \in B_1$ such that

$$\sum_{x \in B_1} \lambda_{ax} x = \rho_a a - \sum_{s \in B_2} \lambda_{as} s.$$

Therefore,

$$\sum_b \lambda_{ab} b = \rho_a a \quad \text{for all} \quad a \in A.$$

Again, using Lemma 3.1.1, we conclude that for all $x \in B_1$,

$$\sum_a \lambda_{ax} a = \gamma_x x.$$

Therefore λ is a stress with the given row and column sums as required. ■

COROLLARY 3.2.4. *Let G be a bipartite framework with a complete bipartite spanning subframework. Order the elements of the set B so that $B_1 = \{b_{p+1}, b_{p+2}, b_{p+3}, ..., b_n\}$. Then $dim(\Omega_G) = \sum_{j=1}^{p} N(A_{b_j}(G)) + dim\mathbb{D}(C_G \otimes C_G)$.*

Notice that in the case where G is a complete bipartite framework $A_{b_j}(G) = A$ for every $b_j \in B_2$ and $p = N(B)$. Hence, $\sum_{j=1}^{p} N(A_{b_j}(G)) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B))$ and

$$C_G = (A \cap < B >) \cup (< A > \cap B) = C.$$

Therefore if G is a complete bipartite framework this theorem gives the same result as the theorem of Bolker and Roth.

Obviously, not every realization of every bipartite framework has the above property. In fact one should note that having a complete bipartite spanning subframework is dependent upon the realization. For example, let G be the framework obtained by removing the two bars, $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from K_{44} realized in the plane. Furthermore assume that the set A has no three points collinear and the set B has no three points collinear. One can easily check that, for this realization, G has no complete bipartite spanning subframework. On the other hand, if G is realized so that B is collinear, then there is a complete bipartite spanning subframework. Simply choose $B_1 = \{b_3, b_4\}$.

Now, we answer the question: Does Corollary 3.2.4 yield the proper dimension of the stress space for frameworks not satisfying its conditions? The next two examples give both a case when a framework G has no complete bipartite spanning subframework and Corollary 3.2.4 does not yield the proper dimension and a case when a different realization of G, still having no complete bipartite spanning subframework, has Corollary 3.2.4 yielding the proper dimension.

As above, let G be a framework obtained by removing the two bars $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from K_{44} realized in the plane. Furthermore let the joints of G be in generic position. First, we note that every joint $b \in B$ is connected, via bars, to at least three joints of the set A and in this realization any three joints of A span the plane. Similarly, every joint of A is connected to at least three joints of the set B. Hence we have $C_G = A \cup B$ and therefore $dim(\mathbb{D}(C_G \otimes C_G)) = 2$. From this we would hope $dim(Im(\tau)_G) = 2$. However, one can check, using the computer, that this realization has $dim(Im(\tau)_G) = 1$.

Now, assume that G is realized, as seen in figure 3.2, where we still have no three points of the set A or of the set B being collinear, the points $\{a_1, a_3, a_4, b_2, b_3, b_4\}$ are on one conic, and the points $\{a_2, a_3, a_4, b_1, b_3, b_4\}$ are on a different conic. Then, as above, we still have $C_G = A \cup B$, $dim(\mathbb{D}(C_G \otimes C_G)) = 2$, and we will now show in this case we have $dim(Im(\tau)_G) = 2$.

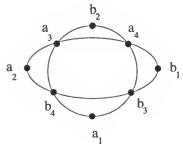


Figure 3.2: Special Realization of K_{44}

THEOREM 3.2.5. *Let G be a framework obtained by removing the two bars $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from K_{44}. Furthermore consider the realization of G where the joints of G lie in the plane as seen in figure 3.2. Then $dim(Im(\tau)_G) = 2$.*

Proof. Choose $\{w_1, w_2\}$ to be a basis for $\mathbb{D}(C_G \otimes C_G)$, where

$$w_1 = (0, x_{a_2}, x_{a_3}, x_{a_4}, x_{b_1}, 0, x_{b_3}, x_{b_4})$$

and $w_2 = (y_{a_1}, 0, y_{a_3}, y_{a_4}, 0, y_{b_2}, y_{b_3}, y_{b_4})$. We have chosen w_1 to be the coefficients of the obvious conic dependence of the set of points $\{a_2, a_3, a_4, b_1, b_3, b_4\}$ and w_2 to be the coefficients of the obvious conic dependence of the set $\{a_1, a_3, a_4, b_2, b_3, b_4\}$. Now we will construct two independent stresses of the framework G, λ^1 and λ^2 such, that $\tau(\lambda^i) = w_i$ for $i = 1, 2$. We will use the technique sketched after Lemma 3.1.3 together with Lemma 3.1.1. Define λ^1 in the following way: For each $a \in A$ assign $\lambda^1_{ab_1} = 0$. Similarly for each $b \in B$ assign $\lambda^1 a_2 b = 0$. Now for $i = 1, 3, 4$ define the remaining elements of λ^1 so that

$$\sum_{j=2}^{4} \lambda^1_{a_i b_j} b_j = x_{a_i} a_i.$$

We can do this since $\{b_2, b_3, b_4\}$ span the plane. Now invoke Lemma 3.1.1 to see that λ^1 is in fact a stress of G with the property that $\tau(\lambda^1) = w_1$. Now similarly, for each $a \in A$ define $\lambda^2_{ab_2} = 0$ and for each $b \in B$ assign $\lambda^2_{a_1 b} = 0$. Now define the remaining entries of λ^2 so that for $i = 1, 3, 4$ we have

$$\sum_{j=2}^{4} \lambda^2_{a_j b_i} a_j = y_{b_i} b_i.$$

Obviously λ^1 and λ^2 are independent, hence we conclude that for this realization of G we have $dim(Im(\tau)_G) = dim(\mathbb{D}(C_G \otimes C_G))$. ∎

Alternatively, there are some frameworks for which Corollary 3.2.4 applies to all but a few very special realizations. For example, if we let $G = K_{44}$ in the plane with only the bar $\{a_1, b_1\}$ removed, then every realization of G with distinct points, such that b_1 is not an isthmus of B, has a complete bipartite spanning subframework.

Therefore we can calculate the dimension of the stress space for G in any of these realizations. For example, if the points of G are in generic position, then $C_G = A \cup B$. By Corollary 3.2.4 if we let $B_1 = \{b_2, b_3, b_4\}$ we know $dim(\Omega_G) = N(A_{b_1}(G)) + dim\mathbb{D}(C_G \otimes C_G) = 0 + 2 = 2$. Using our knowledge of $\mathbb{D}(C_G \otimes C_G)$ and $N(A_{b_1}(G))$ we can use Corollary 3.2.4 to predict the realizations for which Ω_G would change. For instance, if all the points of C_G fall on a circle, then we still have $N(A_{b_1}) = 0$ but now $dim(\mathbb{D}(C_G \otimes C_G)) = 3$.

3.3 Other Realizations

THEOREM 3.3.1. *Let G be a bipartite framework with a complete spanning bipartite subframework. Let G_1 be a framework obtained by removing, from G, any bar $\{a, b\}$ where b is an isthmus of $B_a(G)$. Then*

$$dim(\Omega_{G_1}) = dim(ker(\tau)_G) + dim(\mathbb{D}(C_{G_1} \otimes C_{G_1})).$$

Proof. We begin by showing $dim(Im(\tau)_{G_1}) = dim(\mathbb{D}(C_{G_1} \otimes C_{G_1}))$. Let

$$w = (\rho_a, ..., \gamma_b, ...) \in \mathbb{D}(C_{G_1} \otimes C_{G_1}).$$

We will show that there is $\lambda \in \Omega_{G_1}$ such that $\tau(\lambda) = w$. We know from the definition of C_G that, for each $b \in B_2$ we have $\gamma_b b \in <A_b(G_1)>$. Hence, we can choose coefficients λ_{ab} such that $\sum_a \lambda_{ab} a = \gamma_b b$ with $\lambda_{ab} = 0$ for all $a \notin A_b(G_1)$. Note that, for each $a \in A$, if a bar $\{a, b\}$ was removed from G, then b was an isthmus of $B_a(G)$. Therefore, $B_a(G_1) \cap B_1$ is a basis for $< B_a(G_1) >$. Hence, $\sum_{b \in B_2} \lambda_{ab} b \in < B_a(G_1) \cap B_1 >$. Furthermore, $\rho_a a \in < B_a(G_1) > = < B_a(G_1) \cap B_1 >$ for every $a \in A$. Therefore, we can choose λ_{ab} for $b \in B_a(G_1) \cap B_1$ such that

$$\sum_{b \in B_a \cap B_1} \lambda_{ab} b = \rho_a a - \sum_{b \in B_2} \lambda_{ab} b,$$

which may be rewritten to obtain

$$\sum_{b \in B} \lambda_{ab} b = \rho_a a,$$

where $\lambda_{ab} = 0$ if $b \notin B_a(G_1)$. Now invoking Lemma 3.1.1, we conclude that $\lambda \in \Omega_G$ with the property that $\tau(\lambda) = w \in \mathbb{D}(C_{G_1} \otimes C_{G_1})$. Hence $\mathbb{D}(C_{G_1} \otimes C_{G_1}) \subset Im(\tau)_{G_1}$. Furthermore, from Lemma 3.1.3, we know $Im(\tau)_{G_1} \subset \mathbb{D}(C_{G_1} \otimes C_{G_1})$. Hence,

$$dim(Im(\tau)_{G_1}) = dim(\mathbb{D}(C_{G_1} \otimes C_{G_1})).$$

Next we show $dim(ker(\tau)_{G_1}) = dim(ker(\tau)_G)$. From Lemma 3.1.2 we know $ker(\tau)_{G_1} \subset ker(\tau)_G$. Let $\lambda \in ker(\tau)_G$. Hence every row of $\lambda \in \mathbb{D}(B)$. Now, for each $a \in A$, if $b \notin B_a(G_1)$, and $b \in B_a(G)$, then b is an isthmus of $B_a(G)$. Therefore, b must not be involved in any dependence of $B_a(G)$. Hence, $\lambda_{ab} = 0$. We may now conclude that $\lambda \in ker(\tau)_{G_1}$. Hence every element of $ker(\tau)_G$ is also in $ker(\tau)_{G_1}$ as required. ■

Using this theorem we get the following corollaries about seemingly different realizations.

COROLLARY 3.3.2. *Let G be a bipartite graph with the bars*

$$a_{i_1}b_{j_1}, a_{i_2}b_{j_2}, a_{i_3}b_{j_3}, ..., a_{i_l}b_{j_l}$$

removed from the complete bipartite graph. If $b_{j_1}, b_{j_2}, b_{j_3}, ...b_{j_l}$ are isthmuses, then

$$dim(\Omega_G) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B)) + dim\mathbb{D}(C_G \otimes C_G).$$

Proof. Since $b_{j_1}, b_{j_2}, b_{j_3}, ..., b_{j_l}$ are isthmuses of B, they are also isthmuses of $B_a(G)$ for every $a \in A$. Therefore, by Theorem 3.3.1, we have

$$dim(\Omega_G) = dim(ker(\tau)_{K_{mn}}) + dim\mathbb{D}(C_G \otimes C_G),$$

which is the desired result. ■

COROLLARY 3.3.3. *Let G be a bipartite graph with one of the vertex sets independent. Then $dim(\Omega_G) = dim(\mathbb{D}(C_G \otimes C_G))$*

Proof. Note that every element of the independent set is an isthmus. Hence the result follows immediately from Corollary 3.3.2. ■

COROLLARY 3.3.4. *Let G be a bipartite graph. Let $B_1 = \{b_1, b_2, b_3, ..., b_p\} \subset B$ be a basis of $< B >$. Let G_2 be a framework obtained in the following way. For every $a \in A$ and each $b \in B_1$, if $\{a, b\} \notin G$, remove all the bars of the form $\{a, x\}$ with $x \in B_2 = B - B_1$ from G. Now let G_1 be a framework with only the bars of the form $\{a, b\}$ removed if $\{a, b\} \notin G_2$ and $b \in B_2$ from K_{mn}. Then*

$$dim(\Omega_{G_2}) = dim(ker(\tau)_{G_1}) + dim(\mathbb{D}(C_{G_2} \otimes C_{G_2}))$$

Proof. If we think of obtaining the framework G_2 from G_1 by removing the appropriate bars from G_1, then this is a direct corollary of Theorem 3.3.1. ■

The following corollary calculates $dim(\Omega_G)$ for all realizations of a particular class of frameworks.

COROLLARY 3.3.5. *Let G be the bipartite graph obtained by removing bars from K_{mn} all incident at one vertex $b \in B$.*

(1) If b is an isthmus of B then

$$dim(\Omega_G) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B)) + dim\mathbb{D}(C_G \otimes C_G).$$

(2) Otherwise,

$$dim(\Omega_G) = N(A)N(B) - N(A) + N(A_b(G)) + dim\mathbb{D}(C_G \otimes C_G).$$

Proof. First, assume b is an isthmus of B. Then G satisfies the conditions of Theorem 3.3.1. Since no other bars have been removed, we are done.

Assume now that b is not an isthmus of B. Hence there exists a basis, $B_1 \subset B - \{b\}$, of $< B >$. Therefore G satisfies the conditions of Corollary 3.2.4, from which the desired result follows. We note that in this case for each $x \in B_2 - \{b\}$ we have $A_x(G) = A$. Hence

$$\sum_{x \in B_2} A_x(G) = N(A)N(B) - N(A) + N(A_b(G)).$$

■

There are some interesting frameworks which fall into this class. In particular, let G be a framework obtained by removing the one bar$\{a_1, b_1\}$ from K_{55} realized in 3-space. Techniques of White and Whiteley [16], called pure conditions, will tell us that G has a stress if all the joints of K_{55} fall on a quadric surface. From Corollary 3.3.5 we see that G has a stress if b_1 (or a_1) is an isthmus. Alternatively, if b_1 is not an isthmus, then $dim(ker(\tau)_G) = 0$; and in generic position $dim(\mathbb{D}(C_G \otimes C_G)) = 0$ unless the the joints of K_{55} fall on a quadric surface. But Corollary 3.3.5 gives more information. For instance, if b_1 and a_1 are not isthmuses and if more than 6 points of C_G are coplanar, then there is a stress. Furthermore, for each particular realization of G we can calculate the exact dimension of the stress space of G.

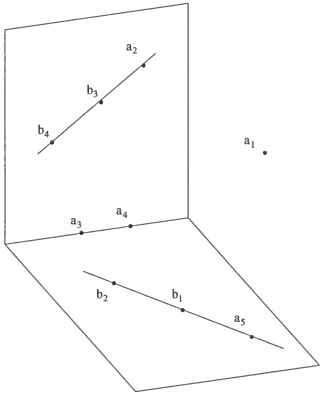


Figure 3.3: Special Realization of K_{54}

The following is an interesting example of a framework G' obtained by removing the bar $\{a_1, b_1\}$ from K_{54}. Choose the following realization. Let the set of points

$\{a_3, a_4, a_2, b_3, b_4\}$ be coplanar and the set $\{a_3, a_4, a_5, b_2, b_1\}$ also be coplanar as seen in figure 3.3. Now choose a_1 so that $[a_1 b_2 b_3 b_4] \neq 0$. In this realization we see that $|C_{G'}| = 8$ and $dim(< C_{G'} >) = 3$, namely the point $a_1 \notin C_{G'}$. In general, for eight points, not coplanar, we have $dim(\mathbb{D}(C_G \otimes C_G)) = 0$, but in this case we will show $dim(\mathbb{D}(C_G \otimes C_G)) = 1$. To see this let x be any other point on the line spanned by a_3 and a_4. Now, clearly the sets $\{b_4, b_3, a_2, a_3, a_4, x\}$ and $\{a_3, a_4, b_2, b_1, a_5, x\}$ are conically dependent in the plane. Hence there are coefficients μ_i such that

$$x \otimes x = \mu_1 b_4 \otimes b_4 + \mu_2 b_3 \otimes b_3 + \mu_3 a_2 \otimes a_2 + \mu_4 a_3 \otimes a_3 + \mu_5 a_4 \otimes a_4.$$

Similarly, there are coefficients η_j such that

$$x \otimes x = \eta_1 a_3 \otimes a_3 + \eta_2 a_4 \otimes a_4 \eta_3 b_2 \otimes b_2 + \eta_4 b_1 \otimes b_1 + \eta_5 a_5 \otimes a_5.$$

Using the above two equations eliminate $x \otimes x$ and acquire a conic dependence on $C_{G'}$. One can show that for this realization $dim(ker(\tau)_G) = 0$. Hence this is an example of a realization of G' which has a stress.

THEOREM 3.3.6. *Let $a_1 \in A$ and $b_1 \in B$ such that a_1 is not an isthmus of A. Let G be a framework obtained by removing the bar $\{a_1, b_1\}$ and any arbitrary set of other bars incident to a_1 or to b_1 from K_{mn}.*

1. If b_1 is not an isthmus of B, then

$$dim(\Omega_G) = dim\mathbb{D}(C_G \otimes C_G) + (N(A) - 1)(N(B) - 1) + N(A_{b_1}(G)) + N(B_{a_1}(G)).$$

2. If b_1 is an isthmus of B, then

$$dim(\Omega_G) = dim\mathbb{D}(C_G \otimes C_G) + N(A)N(B) - N(B) + N(B_{a_1}(G)).$$

Proof. We begin with the following definitions. Let G be obtained by removing the following two sets of bars:

$$\mathcal{B}_1 = \{a_1 b_1, a_1 b_{i_2}, ...a_1 b_{i_k}\}$$

and

$$\mathcal{B}_2 = \{a_{j_2}b_1, a_{j_3}b_1, ..., a_{j_p}b_1\}$$

from K_{mn}. Furthermore define G_1 to be the framework obtained by removing just the set $\mathcal{B}_1$ from K_{mn} and define G_2 to be the framework obtained by removing just the bars of $\mathcal{B}_2$ together with the bar $\{a_1, b_1\}$ from K_{mn}.

First we will prove statement 1. Assume b_1 is not an isthmus. We start with $Im(\tau)_G$. Let $(\rho_a, ..., \gamma_b, ...) \in \mathbb{D}(C_G \otimes C_G)$. Choose a basis $A_1 \subset A - \{a_1\}$ of $< A >$ and a basis $B_1 \subset B - \{b_1\}$ of $< B >$. By the definition of C_G we know, for every $b \in B_2 = B - B_1$ there are coefficients λ_{ab} such that

$$\sum_{a \in A_b(G)} \lambda_{ab} a = \gamma_b b,$$

where we choose $\lambda_{ab} = 0$ if $a \notin A_b(G)$.

First define the entries $\lambda_{a_1 b}$ for $b \in B_1$. Again by the definition of C_G we know there are coefficients $\nu_{a_1 b}$ such that

$$\sum_{b \in B_{a_1}} \nu_{a_1 b} b = \rho_{a_1} a_1.$$

We would like the coefficients $\nu_{a_1 b}$ to be the first row of λ, but $\lambda_{a_1 b}$ for $b \in B_2$ have already been defined. We modify these current values of λ in the following way: Since a_1 is not an isthmus there is a dependence $w \in \mathbb{D}(A)$ such that $w_{a_1} = 1$. Now add the column vector $(\nu_{a_1 b} - \lambda_{a_1 b})w$ to each column indexed by $b \in B_2$. Hence we have $\lambda_{a_1 b} = \nu_{a_1 b}$ for every $b \in B_2$. Finally, for each $b \in B_1$ let $\lambda_{a_1 b} = \nu_{a_1 b}$.

Now, as in previous theorems, we may assign the values λ_{ab} to the remaining undefined entries of λ so that for every $a \in A - \{a_1\}$ we have

$$\sum_{b \in B} \lambda_{ab} b = \rho_a a.$$

Finally, λ satisfies the conditions of Lemma 3.1.1 and may conclude that the $dim(Im(\tau)) = dim\mathbb{D}(C_G \otimes C_G)$.

In calculating $dim(ker(\tau)_G)$, let G_1 and G_2 be denoted as above. Using the definition of $ker(\tau)_G$ we see $ker(\tau)_G = ker(\tau)_{G_1} \cap ker(\tau)_{G_2}$. We note that both G_1 and G_2 satisfy the conditions of Theorem 3.2.2. Hence we can show that

$$dim(ker(\tau)_{G_2}) = N(A)N(B) - N(A) + N(A_{b_1}(G_2))$$

and $dim(ker(\tau)_{G_1}) = N(A)N(B) - N(B) + N(B_{a_1}(G_1))$. In addition, we will show $dim(< (ker(\tau)_{G_1}) + (ker(\tau)_{G_2}) >) = N(A)N(B) - 1$. We start by choosing the following basis for $\mathbb{D}(A) \otimes \mathbb{D}(B)$: For each element $b \in B_2 = B - B_1$ let $w_b = (0, 0, ..., 1, 0, ..., \mu_1, \mu_2, ..., \mu_k)$ where the 1 is in the b^{th} position. Similarly for each $a \in A_2 = A - A_1$ define $e_a = (0, 0, ..., 1, 0, ..., \nu_1, \nu_2, ..., \nu_j)$. Every element $e_a \otimes w_b$ is either in $ker(\tau)_{G_1}$ or $ker(\tau)_{G_2}$ except for the element $e_{a_1} \otimes w_{b_1}$, which has a 1 in the $a_1 b_1$ position. Now, since the set $\{e_a \otimes w_b\}$ is a basis for $\mathbb{D}(A) \otimes \mathbb{D}(B)$ and $< ker(\tau)_{G_1} + ker(\tau)_{G_2} > \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$ we may conclude that

$$dim < ker(\tau)_{G_1} + ker(\tau)_{G_2} >= dim(\mathbb{D}(A) \otimes \mathbb{D}(B)) - 1 = N(A)N(B) - 1.$$

Hence, we find $dim(ker(\tau))_G = (N(A) - 1)(N(B) - 1) + N(A_{b_1}(G_2)) + N(B_{a_1}(G_1))$. Finally, we note that $N(A_{b_1}(G_2)) = N(A_{b_1}(G))$ and $N(B_{a_1}(G_1)) = N(B_{a_1}(G))$, which brings us to the desired result.

For part 2 assume b_1 is an isthmus of B. Except for the row corresponding to a_1, G satisfies the conditions of Theorem 3.3.1. We may use techniques of Theorem 3.3.1 to define all the entries of λ except row a_1. Again, since a_1 is not an isthmus there is a dependence of the set A which has a coefficient of 1 on a_1. We can use this dependence, as in the first part of the proof, to define a λ which satisfies the conditions of Lemma 3.1.1. Therefore we may conclude, in this case also, that $dim(Im(\tau)_G) = dim(\mathbb{D}(C_G \otimes C_G))$.

Now we calculate $dim(ker(\tau)_G)$ for case 2. Use G_1 and G_2 as defined above. Note that Lemma 3.1.2 yields $ker(\tau)_{G_2} \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$. Furthermore, since b_1 is an isthmus, every dependence of the set B is zero on b_1. Hence every element of

$\mathbb{D}(A)\otimes\mathbb{D}(B)$ has zeros in the column corresponding to b_1. Therefore, $\mathbb{D}(A)\otimes\mathbb{D}(B) = ker(\tau)_{G_2}$. Because we also know that $ker(\tau)_{G_1} \subset \mathbb{D}(A)\otimes\mathbb{D}(B)$, we have

$$\begin{aligned} ker(\tau)_G &= ker(\tau)_{G_1} \cap ker(\tau)_{G_2} \\ &= \mathbb{D}(A)\otimes\mathbb{D}(B) \cap ker(\tau)_{G_1} \\ &= ker(\tau)_{G_1} \\ &= N(A)N(B) - N(B) + N(B_{a_1}(G_1)), \end{aligned}$$

which is what we wanted to show. ■

To see how this theorem applies, we take 12 points in generic position in 3-space. Let G be a framework obtained by removing the bars $\{a_1, b_1\}$, $\{a_2, b_1\}$,$\{a_3, b_1\}$, $\{a_1, b_2\}$, $\{a_1, b_3\}$, and $\{a_1, b_4\}$ from K_{66}. Since there are no isthmuses, the above theorem applies. Hence we find $C_G = (A \cup B) - \{a_1, b_1\}$. Thus $dim(\mathbb{D}(C_G \otimes C_G)) = 0$. Now $dim(ker(\tau)_G) = (N(A) - 1)(N(B) - 1) + N(A_{b_1}(G)) + N(B_{a_1}(G)) = 1$. We conclude, in generic position that $dim(\Omega_G) = 1$; and hence the framework is rigid. Furthermore we conclude that if the 10 points of C_G fall on a quadric surface, then $dim(\mathbb{D}(C_G \otimes C_G)) = 1$ and consequently $dim(\Omega_G) = 2$, which yields a framework which is not rigid.

THEOREM 3.3.7. *Let G be a bipartite framework and denote by G_1 a framework obtained by removing one bar from G. If $dim(Im(\tau)_G) = \mathbb{D}(C_G \otimes C_G)$ and $|C_{G_1}| < |C_G|$ then $dim(\Omega_{G_1}) = dim(ker(\tau)_G) + dim\mathbb{D}(C_{G_1} \otimes C_{G_1})$.*

Proof. Assume $dim(Im(\tau)_G) = \mathbb{D}(C_G \otimes C_G)$ and $|C_{G_1}| < |C_G|$ and let $\{a, b\}$ be the bar which has been removed. Without loss of generality assume $a \notin C_{G_1}$ but $a \in C_G$. Since $a \in C_G$ we have $a \in < B_a(G) >$. Now, by assumption since $a \notin C_{G_1}$ we conclude $a \notin < B_a(G) - \{b\} > = < B_a(G_1) >$ hence b is an isthmus of $B_a(G)$.

Let $w \in \mathbb{D}(C_{G_1} \otimes C_{G_1}) \subset \mathbb{D}(C_G \otimes C_G)$. Because $Im(\tau)_G = \mathbb{D}(C_G \otimes C_G)$, there is $\lambda \in \Omega_G$ such that $\tau(\lambda) = w$. Since, $a \notin C_{G_1}$, we know $\rho_a = 0$. Then the a^{th} row of

λ is a dependence of the set $B_a(G)$. Because b is an isthmus of $B_a(G)$, it must be that $\lambda_{ab} = 0$. Therefore $\lambda \in \Omega_G$. Hence for every element $w \in \mathbb{D}(C_{G_1} \otimes C_{G_1})$ there exist $\lambda \in \Omega_{G_1}$ such that $\tau(\lambda) = w$. Therefore we may conclude $Im(\tau)_{G_1} = D(C_{G_1} \otimes C_{G_1})$.

A similar argument works for showing $ker(\tau)_{G_1} = ker(\tau)_G$. It is obvious that $ker(\tau)_{G_1} \subset ker(\tau)_G$. Let a and b be as stated above and let $\lambda \in ker(\tau)_G$. Again, the a^{th} row of λ is a dependence of the set $B_a(G)$. Therefore, as above, $\lambda_{ab} = 0$. Hence $\lambda \in ker(\tau)_{G_1}$. Thus we have shown the reverse inclusion and that $ker(\tau)_{G_1} = ker(\tau)_G$. ∎

One specific point Theorem 3.3.7 brings up is a way to go from one framework G where $Im(\tau)_G = \mathbb{D}(C_G \otimes C_G)$ to another framework G' where $Im(\tau)_{G'} = \mathbb{D}(C_{G'} \otimes C_{G'})$ by removing a specific bar. It also raises the following question: Given a framework G which does not have the property that $Im(\tau)_G = \mathbb{D}(C_G \otimes C_G)$ can we remove a bar or bars so that the resulting framework G' is non-trivial and has the property $Im(\tau)'_G = \mathbb{D}(C'_G \otimes C'_G)$? We answer this question with an example. Let G be the framework looked at earlier which is obtained by removing the two bars $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from the complete bipartite framework K_{44} realized in the plane in generic position. We have already noted that for this realization $Im(\tau)_G \neq \mathbb{D}(C_G \otimes C_G)$. Yet, if we remove one extra bar $\{a_2, b_1\}$ to obtain the framework we call G' we see that G' satisfies the hypotheses of Corollary 3.3.4. Hence $Im(\tau)_{G'} = \mathbb{D}(C_{G'} \otimes C_{G'})$.

THEOREM 3.3.8. *Let G be obtained from K_{mn} by removing the bar $\{a, b\}$. Then*

$$dim(ker(\tau)_G) = dim(ker(\tau)_{K_{mn}})$$

if and only if either a is an isthmus of A or b is an isthmus of B.

Proof. Assume $ker(\tau)_G = ker(\tau)_{K_{mn}} = \mathbb{D}(A) \otimes \mathbb{D}(B)$. Then for every $\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ we have $\lambda_{ab} = 0$. We show this implies that one of a or b is an isthmus. Assume a is not an isthmus of A and b is not an isthmus of B. Then there exists $w \in \mathbb{D}(A)$ with $w_a \neq 0$ and there exists $e \in \mathbb{D}(B)$ with $e_b \neq 0$. Consider $w \otimes e \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ and note that $(w \otimes e)_{ab} \neq 0$, a contradiction.

Conversely, without loss of generality, assume that a is an isthmus. Then every element $w \in \mathbb{D}(A)$ has $w_a = 0$; hence every element $\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ has $\lambda_{ab} = 0$. Therefore $ker(\tau)_G = ker(\tau)_{K_{mn}}$. ■

THEOREM 3.3.9. *Let G be obtained from K_{mn} by removing the one bar $\{a, b\}$. Then $dim(Im(\tau)_G) = dim(Im(\tau)_{K_{mn}})$ if and only if either*

(1) $C_G = C_{K_{mn}}$

or

(2) $|C_G| < |C_{K_{mn}}|$ and the elements of $C_{K_{mn}} - C_G$ are conic isthmuses.

Proof. Using Corollary 3.3.5 we know $dim(Im(\tau)_G) = dim(\mathbb{D}(C_G \otimes C_G))$. Furthermore, using Bolker and Roth's result we know

$$dim(Im(\tau)_{K_{mn}}) = dim(\mathbb{D}(C \otimes C)).$$

Therefore, $dim(Im(\tau)_G) = dim(Im(\tau)_{K_{mn}})$ if and only if

$$dim(\mathbb{D}(C_G \otimes C_G)) = dim(\mathbb{D}(C_{K_{mn}} \otimes C_{K_{mn}})).$$

Now, since $C_G \subset C_{K_{mn}}$, we know that $\mathbb{D}(C_G \otimes C_G) \subset \mathbb{D}(C_{K_{mn}} \otimes C_{K_{mn}})$. We get equality if and only if either $|C_G| = |C_{K_{mn}}|$ or the points of $C_{K_{mn}} - C_G$ are not involved in any dependence and hence are conic isthmuses. Therefore the theorem is proved. ■

The following two similar realizations of the framework G obtained by removing the one bar $\{a_1, b_1\}$ from the framework K_{43} realized in the plane illustrates how Theorem 3.3.9 identifies redundant bars. First, let G be realized as seen in figure 3.4. Note that $b_1 \notin < A_{b_1}(G) >$, hence $b_1 \notin C_G$. Furthermore, notice that if $C = A \cup B$ then $dim(\mathbb{D}(C \otimes C)) = 1$. We also see that $C_G = \{a_1, a_2, a_3, a_4, b_2, b_3\} \subset C$ falls on a conic hence $dim(\mathbb{D}(C_G \otimes C_G)) = 1$ hence b_1 is a conic isthmus. Therefore Theorem 3.3.9 tells us that, for this realization, we can remove any bar incident to b_1 whose removal changes C_G by exactly the point b_1 and not change the dimension of $Im(\tau)_G$.

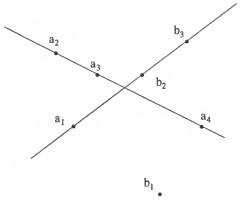


Figure 3.4: Special Realization of K_{43}

Rather, notice if we select the realization seen in figure 3.5 we obtain a very similar looking framework G. We have only interchanged a_1 and a_4 which did not change $dim(< A >)$ or $dim(< B >)$. Moreover observe it is a_1 which in not in the

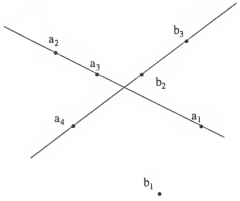


Figure 3.5: Special Realization of K_{43}

span of $B_{a_1}(G)$, hence $C_G = \{a_2, a_3, a_4, b_1, b_2, b_3\}$. Therefore C_G consists of 6 points in the plane, which are not on a conic, thus not conically dependent. Accordingly we find $dim(Im(\tau)_G) = dim(\mathbb{D}(C_G \otimes C_G)) = 0$.

THEOREM 3.3.10. *Let G_1 be the framework obtained by removing the bar $\{a_1, b_1\}$ from K_{mn} and let G_2 be the framework obtained by removing the bar $\{a_2, b_2\}$ from G_1. Then $dim(ker(\tau)_{G_1}) = dim(ker(\tau)_{G_2})$ if and only if at least one of the following happens:*

1. $ker(\tau)_{G_1}$ is trivial.

2. a_2 or b_2 is an isthmus.

3. Neither a_1 nor b_1 is an isthmus, $A-\{a_1,a_2\}$ is contained in a hyperplane of $< A >$, and $B-\{b_1,b_2\}$ is contained in a hyper-plane of $< B >$.

Proof. Assume $ker(\tau)_{G_1} = ker(\tau)_{G_2}$. This would happen if $ker(\tau)_{G_1} = \{0\}$. Hence assume $ker(\tau)_{G_1} \neq \{0\}$. We know that every $\lambda \in ker(\tau)_{G_1}$ has $\lambda_{a_1b_1} = 0$. We also know, by the above assumption, that every $\lambda \in ker(\tau)_{G_1}$ also has $\lambda_{a_2b_2} = 0$. Clearly this would happen if a_2 was an isthmus of A or b_2 is an isthmus of B.

Now assume neither a_2 nor b_2 are isthmuses of A and B respectively and $ker(\tau)_{G_1}$ is not trivial. First we show, in this case, that neither a_1 nor b_1 are isthmuses. Assume a_1 is an isthmus. Since neither a_2 nor b_2 are isthmuses one can easily construct an element $\lambda \in ker(\tau)_{G_1}$ which has the property that $\lambda_{a_2b_2} \neq 0$. Hence $\lambda \notin ker(\tau)_{G_2}$ which contradicts our original assumption. Thus we may conclude that a_1 is not an isthmus. A similar argument will show b_1 is not an isthmus. Therefore we are in the case when $ker(\tau)_{G_1}$ is not trivial and none of a_2, b_2, a_1, nor b_1 are isthmuses. Hence we need only to show $A-\{a_1,a_2\}$ is contained in a hyperplane of $< A >$ and $B-\{b_1,b_2\}$ is contained in a hyperplane of $< B >$.

We show this by way of contradiction. Assume, without loss of generality that $A-\{a_1,a_2\}$ is not contained in a hyperplane of $< A >$. Hence there is a basis $A_1 \subset A$ of $< A >$ which does not contain either of the points a_1 or a_2. Therefore, there exists an element $w \in \mathbb{D}(A)$ such that $w_{a_1} = 0$ and $w_{a_2} = 1$. Furthermore, since b_2 is not an isthmus there exists an element $e \in \mathbb{D}(B)$ such that $e_{b_2} \neq 0$. Hence, $\lambda = w \otimes e$ has the property that $\lambda_{a_1b_1} = 0$ so $\lambda \in ker(\tau)_{G_1}$. Moreover, $\lambda_{a_2b_2} \neq 0$ consequently $\lambda \notin ker(\tau)_{G_2}$, which contradicts the assumption that $ker(\tau)_{G_1} = ker(\tau)_{G_2}$.

Conversely, the conclusion obviously follows if $ker(\tau)_{G_1}$ is trivial. On the other hand, assume $ker(\tau)_{G_1}$ is not trivial and a_2 is an isthmus. Then for every element $\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B)$ we have $\lambda_{a_2b_2} = 0$. Since $ker(\tau)_{G_1} \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$ then every

element of $\lambda \in ker(\tau)_{G_1}$ has $\lambda_{a_2b_2} = 0$. Thus $ker(\tau)_{G_1} = ker(\tau)_{G_2}$. The result follows similarly in the case when b_2 is an isthmus.

Now for the final case, assume that $ker(\tau)_{G_1}$ is not trivial and neither a_2 nor b_2 are isthmuses. Furthermore, assume that neither a_1 nor b_1 are isthmuses, $A - \{a_1, a_2\}$ is contained in a hyperplane of $< A >$ and $B - \{b_1, b_2\}$ is contained in a hyperplane of $< B >$. Hence, we my pick a basis $A_1 \subset A$ for $< A >$ which includes a_2 and not a_1. Similarly, we can pick a basis $B_1 \subset B$ for $< B >$ which includes b_2 and not b_1. Now we can choose a basis $\mathcal{A}$ of $\mathbb{D}(A)$ so that there is exactly one element $w \in \mathbb{D}(A)$ with both $w_{a_1} \neq 0$ and $w_{a_2} \neq 0$ namely a dependence which comes from writing a_1 as a linear combination of the elements of A_1. Because $A - \{a_1, a_2\}$ is contained in a hyperplane of $< A >$, we note that for every $\eta \in \mathcal{A} - \{w\}$, we have $\eta_{a_1} = 0$ and $\eta_{a_2} = 0$. Likewise, we can find a similar basis $\mathcal{B}$ of $\mathbb{D}(B)$. The set $\mathcal{A} \otimes \mathcal{B}$ is a basis for $\mathbb{D}(A) \otimes \mathbb{D}(B)$ which only has one element λ which has $\lambda_{a_1b_1} \neq 0$ and $\lambda_{a_2b_2} \neq 0$. Furthermore, every element $\mu \in \mathcal{A} \otimes \mathcal{B} - \{\lambda\}$ has $\mu_{a_1b_1} = 0$ and $\mu_{a_2b_2} = 0$. Therefore $\mu \in ker(\tau)_{G_1}$ and $\mu \in ker(\tau)_{G_2}$. Hence we conclude, in this final case, $ker(\tau)_{G_1} = ker(\tau)_{G_2}$. ∎

THEOREM 3.3.11. *Let G_1 and G_2 be denoted as above in Theorem 3.3.10, then $Im(\tau)_{G_2} = Im(\tau)_{G_1}$ if and only if*

1. $Im(\tau)_{G_2} = \mathbb{D}(C_{G_2} \otimes C_{G_2})$ *and*
2. $\mathbb{D}(C_{G_1} \otimes C_{G_1}) = \mathbb{D}(C_{G_2} \otimes C_{G_2})$.

Proof. Assume $Im(\tau)_{G_1} = Im(\tau)_{G_2}$. Furthermore, assume that

$$Im(\tau)_{G_2} \neq \mathbb{D}(C_{G_2} \otimes C_{G_2}).$$

Now $dim(\mathbb{D}(C_{G_2} \otimes C_{G_2})) \leq dim(\mathbb{D}(C_{G_1} \otimes C_{G_1}))$ and $Im(\tau)_{G_1} = \mathbb{D}(C_{G_1} \otimes C_{G_1})$. Since, by Lemma 3.1.3, we know $Im(\tau)_{G_2} \subset \mathbb{D}(C_{G_2} \otimes C_{G_2})$ we have,

$$dim(Im(\tau)_{G_2}) \quad < dim(\mathbb{D}(C_{G_2} \otimes C_{G_2}))$$

$$\leq dim(\mathbb{D}(C_{G_1} \otimes C_{G_1}))$$

$$= dim(Im(\tau)_{G_1}$$

which is a contradiction. Hence, it must be that $Im(\tau)_{G_2} = \mathbb{D}(C_{G_2} \otimes C_{G_2})$. Now condition 2 is obvious, as is the converse. ■

CHAPTER 4
GENERAL RESULTS

4.1 The Kernel

In general, we would like to answer the following question: Given any realization of a bipartite framework G what is the dimension of Ω_G? To this end, we restate Lemma 3.1.2 and Lemma 3.1.3. We know for any realization of a bipartite framework G that $ker(\tau)_G \subset \mathbb{D}(A) \otimes \mathbb{D}(B)$ and $Im(\tau)_G \subset \mathbb{D}(C_G \otimes C_G)$. In fact

$$ker(\tau)_G = \{\lambda \in \mathbb{D}(A) \otimes \mathbb{D}(B) | \lambda_{ab} = 0 \text{ if } \{a, b\} \notin G\}.$$

We try, again, to tackle this problem in two parts. We begin by looking at $ker(\tau)_G$.

First, we use use Theorem 3.2.2 to get a slightly better upper bound on $dim(ker(\tau)_G)$.

THEOREM 4.1.1. *Let G be a bipartite framework which is obtained by removing l bars from the complete bipartite framework K_{mn}. Furthermore, choose a basis $B_1 = \{b_{p+1}, b_{p+2}, b_{p+3}, ..., b_n\} \subset B$ for $< B >$. Define G' to be a framework obtained from K_{mn} by removing the same bars removed to obtain G, except those of the form $\{a, b\}$, where $b \in B_1$. Then*

$$dim(ker(\tau)_G) \leq \sum_{j=1}^{p} N(A_{b_j}(G')).$$

Proof. Clearly $ker(\tau)_G \subset ker(\tau)_{G'}$. Now we simply use the formula from Theorem 3.2.2 to reach the desired result. ∎

At this point, it will be to our advantage to write $s \otimes t$ in the following way: Given an m-tuple s and an n-tuple t we will write $s \otimes t$ as an mn-tuple in some fixed order.

THEOREM 4.1.2. *Let G be a framework, which is obtained by removing the set*

$$\mathcal{B} = \{\{a_{i_1}, b_{j_1}\}, \{a_{i_2}, b_{j_2}\}, \{a_{i_3}, b_{j_3}\}, ..., \{a_{i_l}, b_{j_l}\}\}$$

of l bars from K_{mn}. Then $dim(ker(\tau)_G) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B)) - l$ if and only if there exist a basis $\mathcal{D} = \{w_1, w_2, w_3, ..., w_l, w_{l+1}, ..., w_p\}$ of $\mathbb{D}(A) \otimes \mathbb{D}(B)$ such that for each $k = 1, 2, 3, ..., l$ we have a one in the $a_{i_k} b_{j_k}$ position of w_k and a zero in the positions of w_k associated with the bars of $\mathcal{B} - \{\{a_{i_k}, b_{j_k}\}\}$.

Proof. First assume there exist a basis $\mathcal{D}$ with the property stated above. Since we know that $ker(\tau)_G$ is the subspace of $\mathbb{D}(A) \otimes \mathbb{D}(B)$ with zeros in the entries associated with the bars of $\mathcal{B}$ we see that $ker(\tau)_G \cap \mathcal{D} = \emptyset$. Hence

$$dim(ker(\tau_G)) \leq dim(\mathbb{D}(A) \otimes \mathbb{D}(B)) - l.$$

Now, clearly, for each element $w_k \in \mathcal{D}$ with $k = 1, 2, 3, ..., p - l$ there are coefficients $\alpha_{1_k}, \alpha_{2_k}, \alpha_{3_k}, ..., \alpha_{l_k}$ such that

$$n_k = w_{l+k} - \sum_{t=1}^{l} \alpha_{t_k} w_t \in ker(\tau)_G.$$

We claim $N = \{n_1, n_2, n_3, ...n_{p-l}\}$ is independent. Assume there are coefficients $\nu_1, \nu_2, \nu_3, ..., \nu_{p-l}$ such that

$$\sum_{k=1}^{p-l} \nu_k n_k = 0.$$

Therefore we would have

$$0 = \sum_{k=1}^{p-l} \nu_k (w_{l+k} - \sum_{t=1}^{l} \alpha_{t_k} w_t).$$

Hence, expanding the two summations and distributing the ν_k's we see this reduces to

$$0 = \sum_{k=1}^{p-l} \nu_k w_{l+k} - \sum_{k=1}^{p-l} \sum_{t=1}^{l} \nu_k \alpha_{t_k} w_t.$$

Now, if we switch the order of summation and write

$$\sum_{k=1}^{p-l} \nu_k \alpha_{t_k} = \zeta_t$$

we have,

$$0 = \sum_{k=1}^{p-l} \nu_k w_{l+k} - \sum_{t=1}^{l} \zeta_t w_t.$$

Since $\{w_1, w_2, ...w_p\}$ is independent we find for each $k = 1, 2, 3, ..., p-l$ we have $\nu_k = 0$. From this we conclude N is independent.

Conversely, choose a basis $\mathcal{D} = \{w_1, w_2, w_3, ..., w_p\}$ of $\mathbb{D}(A) \otimes \mathbb{D}(B)$. Let M be the matrix whose columns are the coordinates of the elements of $\mathcal{D}$. Hence M has its columns indexed by the elements of $\mathcal{D}$ and its rows indexed by the bars of the complete bipartite framework K_{mn}. Order the rows so that the elements of $\mathcal{B}$ are the first l rows. Since we know, by assumption, removing the l bars of $\mathcal{B}$ causes the dimension of $\mathbb{D}(A) \otimes \mathbb{D}(B)$ to decrease by l, then there exists a nonzero $l \times l$ minor among its first l rows of M. Choose the l columns corresponding to the nonzero minor and column reduce them to obtain the appropriate basis. ■

Starting with a bipartite framework G, which is obtained from the complete bipartite framework K_{mn} by removing l bars $e_1, e_2, e_3, ..., e_l$ from K_{mn} we would like to answer the question: When does there exist a basis of $\mathbb{D}(A) \otimes \mathbb{D}(B)$ which satisfies the conditions of Theorem 4.1.2?

In the previous proof we constructed a matrix M whose columns are the coordinates of the elements of a basis for $\mathbb{D}(A) \otimes \mathbb{D}(B)$. To attempt to answer the above question we will need some more notation. Let $p_A = dim(\mathbb{D}(A))$ and

$$P = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & ... & \alpha_{1p_A} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & ... & \alpha_{2p_A} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & ... & \alpha_{3p_A} \\ . & & & & \\ . & . & & & \\ . & . & . & & \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & ... & \alpha_{mp_A} \end{pmatrix}$$

where the columns of P form a basis for $\mathbb{D}(A)$. Similarly, let $p_B = dim(\mathbb{D}(B))$ and

$$Q = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \dots & \beta_{1p_B} \\ \beta_{21} & \beta_{22} & \beta_{23} & \dots & \beta_{2p_B} \\ \beta_{31} & \beta_{32} & \beta_{33} & \dots & \beta_{3p_B} \\ . & & & & \\ . & . & & & \\ . & . & . & & \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & \beta_{np_B} \end{pmatrix}$$

where the columns of Q form a basis for $\mathbb{D}(B)$. Now, define a matrix M whose rows are indexed by the bars and whose columns are indexed by ordered pairs of columns of P and Q. We choose the following natural lexocographical ordering on the columns of M:

$$(1,1)(1,2)(1,3)...(1,p_B)(2,1)(2,2)(2,3)...(2,p_B)...(p_A,1)(p_A,2)(p_A,3)...(p_A,p_B)$$

Now let the entries of the $(i,j)^{th}$ column of M are defined to be the coordinates of $C_i(P) \otimes C_j(Q)$. Choose a similar ordering on the rows of M.

The columns of M form a basis for $\mathbb{D}(A) \otimes \mathbb{D}(B)$ and doing any of the matrix column operations will not change this. Therefore our question reduces to: When does the $l \times p$ submatrix of M composed of the rows of M which correspond to the l bars, which are removed from K_{mn} to form the framework G, have full rank?

We can see from the construction of M that every column of M is a tensor product of a column from P with a column from Q. Hence we start with the following lemma.

LEMMA 4.1.3. *Let P,Q, and M be the matrices mentioned above. Every row of M is the tensor product of a row from P with a row from Q.*

Proof. We know each column M is a tensor product of a particular column of P with a particular column from Q. Now let M' be a matrix indexed as M where the $\{a_i, b_j\}^{th}$ row of M' is given by the coordinates of $R_{a_i}(P) \otimes R_{b_j}(Q)$. Here we mean $R_{b_j}(P)$ to be the row of Q associated with the point b_j. Now we compare $M_{\{a_i b_j\}(q,r)}$

with $M'_{\{a_ib_j\}(q,r)}$ and see that

$$M_{\{a_ib_j\}(q,r)} = M'_{\{a_ib_j\}(q,r)} = \alpha_{ij}\beta_{qr}$$

This shows that $M = M'$, which was our claim. ■

We know that the first l rows of M have rank l if and only if the first l rows are independent. Hence we would now like to answer the question: When is a set of tensor products independent? In particular, let $T = \{x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3, ..., x_n \otimes y_n\}$ with $X = \{x_1, x_2, x_3, ..., x_n\} \subset \mathbb{P}^d$ a multiset and $Y = \{y_1, y_2, y_3, ..., y_n\} \subset \mathbb{P}^d$ such that $dim < X >= r$ and $dim < Y >= q$. Clearly, if $n > rq$, we know T is dependent. The converse is not necessarily true. To show this, we give an example when $q = r$ and $n = r + r$. Note that if $r > 2$, then $r + r < r^2$. Hence we will have an example T is dependent yet $n < rq$. Let T be as above with $dim(\{x_1, x_2, x_3, ...x_{r+1}\}) = r$ with x_{r+1} not an isthmus of X. Similarly, let $dim(\{y_1, y_2, y_3, ..., y_{r+1}\})$ with y_{r+1} not an isthmus of Y. Then obviously, the set $\{x_{r+1} \otimes y_1, x_{r+1} \otimes y_2, x_{r+1} \otimes y_3, ..., x_{r+1} \otimes y_{r+1}\}$ is dependent. Furthermore the set $\{x_1 \otimes y_{r+1}, x_2 \otimes y_{r+1}, x_3 \otimes y_{r+1}, ..., x_{r+1} \otimes y_{r+1}\}$ is also dependent. Hence there are coefficients η_i and η_i such that

$$x_{r+1} \otimes y_{r+1} = \sum_{i=1}^{r} \nu_i x_i \otimes y_{r+1}$$

and

$$x_{r+1} \otimes y_{r+1} = \sum_{i=1}^{r} \eta_i x_{r+1} \otimes y_i$$

hence

$$\sum_{i=1}^{r} \nu_i x_i \otimes y_{r+1} = \sum_{i=1}^{r} \eta_i x_{r+1} \otimes y_i.$$

Now we show that if $n < r + q$ then T must be independent.

THEOREM 4.1.4. *Let* $T = \{x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3, ..., x_n \otimes y_n\}$ *with*

$X = \{x_1, x_2, x_3, ..., x_n\} \subset \mathbb{P}^d$ *and*

$Y = \{y_1, y_2, y_3, ..., y_n\} \subset \mathbb{P}^d$

both multisets such that dim $< X >= r$ and dim $< Y >= q$. If $n < r + q$ then T is independent.

Proof. Let T, X, and Y be denoted as above and assume, without loss of generality, that T is minimally dependent and that $n < r+q$. Furthermore arrange the elements of T such that $y_1, y_2, y_3, ..., y_q$ form a basis of $< Y >$. Since T is minimally dependent, there exist coeffiecients ν_i such that

$$\nu_1 x_1 \otimes y_1 + \nu_2 x_2 \otimes y_2 + \nu_3 x_3 \otimes y_3 + ... + \nu_{r+q} x_n \otimes y_n = 0 \tag{4.1}$$

and $\nu_i \neq 0$ for all $i = 1, 2, 3, ..., n$. For $s = q+1, q+2, q+3, ..., n$ write $y_s = \sum_{i=1}^{q} \gamma_{s,i} y_i$. This will allow us to rewrite equation 4.1 in the following way:

$$\begin{aligned} 0 = \ & \nu_1 x_1 \otimes y_1 + \nu_2 x_2 \otimes y_2 + \nu_3 x_3 \otimes y_3 + ... + \nu_q x_q \otimes y_q + \\ & \nu_{q+1} x_{q+1} \otimes \textstyle\sum_{t=1}^{q} \gamma_{q+1,t} y_t + ... \nu_n x_n \otimes \sum_{t=1}^{q} \gamma_{n,t} y_t. \end{aligned}$$

Using linearity of the tensor product and grouping like terms we find

$$\begin{aligned} 0 = \ & (\nu_1 x_1 + \textstyle\sum_{s=q+1}^{n} \nu_s \gamma_{s1} x_1) \otimes y_1 + (\nu_2 x_2 + \sum_{s=q+1}^{n} \nu_s \gamma_{s2} x_2) \otimes y_2 + ... + \\ & (\nu_q x_q + \textstyle\sum_{s=q+1}^{n} \nu_s \gamma_{sq} x_q) \otimes y_q. \end{aligned}$$

Because $\{y_1, y_2, y_3, ..., y_q\}$ is independent we may conclude [7] that for $t = 1, 2, 3, ..., q$ we have

$$\nu_t x_t + \sum_{s=q+1}^{n} \nu_s \gamma_{st} x_t = 0.$$

This implies that every $x_i \in X$ is a linear combination of $x_{q+1}, x_{q+2}, x_{q+3}, ..., x_n$. Since we have assumed $n < r + q$, this implies that every element $x_i \in X$ can be written as a linear combination of $n - q$ elements, which is fewer than r elements, a contradiction. ∎

Thus, in general, we may state the following corollary.

COROLLARY 4.1.5. *Let $T = \{x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3, ..., x_n \otimes y_n\}$ with*

$X = \{x_1, x_2, x_3, ..., x_n\} \subset \mathbb{P}^d$ and

$Y = \{y_1, y_2, y_3, ..., y_n\} \subset \mathbb{P}^d$

both multisets such that $dim < X >= r$ *and* $dim < Y >= q$.

1. *If* $n > rq$ *then* T *is dependent.*

2. *If* $n < r + q$ *then* T *is independent.*

Corollary 4.1.5 does not cover the cases when $r + q \leq n \leq rq$, which is a difficult problem. The following theorem addresses these cases.

THEOREM 4.1.6. *Let* T, X, *and* Y *be denoted as in Corollary 4.1.5. The set* T *is dependent if and only if the following are true:*

1. $n \geq r + q$

2. *There is an ordering of the elements of* T *such that* $\{y_1, y_2, y_3, ..., y_q\}$ *is a basis for* $< Y >$, *and the set* $\{x_{q+1}, x_{q+2}, x_{q+3}, ..., x_n\}$ *is a spanning set for* $< X >$.

3. *There exist* $q \times (n - q)$ *coefficients* δ_{ij} *such that*

$$\sum_{j=1}^{n-q} \delta_{ij} x_{q+j} = c_i x_i \quad \text{for some} \quad c_i \quad \text{and for all} \quad i = 1, 2, 3, ..., q,$$

and

$$\sum_{i=1}^{q} \delta_{ij} y_i = w_j y_j \quad \text{for some} \quad w_i \quad \text{and for all} \quad j = q+1, q+2, q+3, ..., n.$$

Proof. Assume first that T is minimally dependent and that we have already ordered the elements of T so that $\{y_1, y_2, y_3, ..., y_q\}$ is a basis for Y. Therefore we know from Corollary 4.1.5 that $n \geq r + q$. Moreover, there are coefficients α_i such that $\sum_{i=1}^{n} \alpha_i (x_i \otimes y_i) = 0$. There also exist coefficients such that for each $j = 1, 2, 3, ..., n-q$ we have

$$y_{q+j} = \sum_{k=1}^{q} \gamma_{jk} y_k. \tag{4.2}$$

Now, following the same steps as in Theorem 4.1.4, we may rewrite the original dependence on T in the following way:

$$\begin{aligned} 0 = \; & (\alpha_1 x_1 + \textstyle\sum_{j=1}^{n-q} \alpha_j \gamma_{ji} x_{q+j}) \otimes y_1 + (\alpha_2 x_2 + \sum_{j=1}^{n-q} \alpha_j \gamma_{ji} x_{q+j}) \otimes y_1 + \\ & (\alpha_3 x_3 + \textstyle\sum_{j=1}^{n-q} \alpha_j \gamma_{ji} x_{q+j}) \otimes y_1 + \ldots + \\ & (\alpha_q x_q + \textstyle\sum_{j=1}^{n-q} \alpha_j \gamma_{ji} x_{q+j}) \otimes y_q. \end{aligned}$$

Now, as before, since $\{y_1, y_2, y_3, ..., y_q\}$ is independent, we conclude for $i = 1, 2, 3, ..., q$ we have

$$\alpha_i x_i + \sum_{j=1}^{n-q} \alpha_j \gamma_{ij} x_{q+j} = 0. \tag{4.3}$$

Equation 4.3 shows that the set $\{x_{q+1}, x_{q+1}, x_{q+1}, ..., x_n\}$ is a spanning set for $< X >$. Furthermore, let $\delta_{ij} = \alpha_j \gamma_{ji}$ and we see

$$\sum_{i=1}^{q} \delta_{ij} x_i = \sum_{i=1}^{q} \alpha_j \gamma_{ji} x_i = \alpha_j x_j, \tag{4.4}$$

which, together with equation 4.2, gives us the desired result.

Conversely, assume conditions 1, 2, and 3 are true. Since $\sum_{j=1}^{n-q} \delta_{ij} x_j = c_i x_i$ for $i = 1, 2, 3, ..., q$ we know

$$(\sum_{i=1}^{q} -c_i x_i + \sum_{j=1}^{n-q} \alpha_j \gamma_{ji} x_{q+j}) \otimes y_i = 0.$$

Since $\otimes$ is linear, we rewrite the above as

$$\sum_{i=1}^{q} c_i x_i \otimes y_i + \sum_{i=1}^{q} \sum_{j=1}^{n-q} \delta_{ij} x_{q+j} \otimes y_i = 0.$$

Now, switching the order of summation, we find

$$\sum_{i=1}^{q} c_i x_i \otimes y_i + \sum_{j=1}^{n-q} x_{q+j} \otimes \sum_{i=1}^{q} \delta_{ij} y_i = 0.$$

Using the assumptions, the above becomes

$$\sum_{i=1}^{q} c_i x_i \otimes y_i + \sum_{j=1}^{n-q} x_{q+j} \otimes w_j y_j = 0,$$

from which we conclude that the set T is dependent and this completes the proof. ■

In general, this theorem is very difficult to use. We have included it here because, in the particular case when $r = q = 2$, we can use this theorem to answer the question completely. When $r = q = 2$ we have $rq = r + q = 4$. Thus we would know exactly when T was independent by knowing what happens when $n = 4$.

THEOREM 4.1.7. *Let* $T = \{x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3, x_4 \otimes y_4\}$, $X = \{x_1, x_2, x_3, x_4\}$ *and* $Y = \{y_1, y_2, y_3, y_4\}$. *We reemphasize here that* X *and* Y *are multisets. Furthermore, we are assuming* T *is ordered so that* y_1 *and* y_2 *span* $< Y >$, *and* x_3 *and* x_4 *span* $< X >$. T *is dependent if and only if exactly one of the following occurs.*

1. *At least one of* $[x_1, x_3], [x_2, x_4], [y_1, y_3], [y_2, y_4]$ *is zero,*

or

2. *The cross ratios of the two sets of four points are equal, that is*

$$\frac{[x_1x_4][x_2x_3]}{[x_1x_3][x_2x_4]} = \frac{[y_1y_4][y_2y_3]}{[y_1y_3][y_2y_4]}. \tag{4.5}$$

The expression on either side of the equal sign of equation 4.5 is referred to as the *cross ratio* of the four collinear points. From projective geometry we know that two sets of four collinear points, one lying on line l_1 and the other set lying on line l_2, have the same cross ratio if and only if there is a projective transformation which takes l_1 to l_2 in such a way that the transformation takes the one set of four points to the other set of four points [13].

Proof. We know Y is arranged so that $\{y_1, y_2\}$ is a basis for $< Y >$. Hence there are coefficients γ_{jk} such that

$$y_3 = \gamma_{11}y_1 + \gamma_{12}y_2 \tag{4.6}$$

and

$$y_4 = \gamma_{21}y_1 + \gamma_{22}y_2. \tag{4.7}$$

If we join every term in equation 4.6 with y_2 and then with y_1 we find

$$\gamma_{11} = \frac{[y_3y_2]}{[y_1y_2]} \quad \text{and} \quad \gamma_{12} = \frac{[y_3y_1]}{[y_2y_1]}. \tag{4.8}$$

Similarly from equation 4.7 we find

$$\gamma_{21} = \frac{[y_4 y_2]}{[y_1 y_2]} \text{ and } \gamma_{22} = \frac{[y_4 y_1]}{[y_2 y_1]}. \tag{4.9}$$

We know that T is dependent if and only if there exists a 2×2 matrix which satisfies condition 3 of Theorem 4.1.6. This is equivalent to determining when there exist coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_3 \gamma_{11} x_3 + \alpha_4 \gamma_{12} x_4 = -\alpha_1 x_1 \tag{4.10}$$

and

$$\alpha_3 \gamma_{21} x_3 + \alpha_4 \gamma_{22} x_4 = -\alpha_2 x_2. \tag{4.11}$$

We see this is true because the matrix

$$\begin{pmatrix} \alpha_3\gamma_{11} & \alpha_4\gamma_{12} \\ \alpha_3\gamma_{21} & \alpha_4\gamma_{22} \end{pmatrix}$$

satisfies condition 3 of Theorem 4.1.6 and one can easily see any matrix satisfying condition 3 of Theorem 4.1.6 can be put in this form.

We answer the question of the existence of α_1, α_2, α_3, and α_4 by looking at the following system of four equations and four unknowns obtained by first joining every term in equation 4.10 with x_1 and then x_3 to find the following two equations:

$$\alpha_3 \gamma_{11} [x_3 x_1] + \alpha_4 \gamma_{12} [x_4 x_1] = 0$$

and

$$\alpha_4 \gamma_{12} [x_4 x_3] + \alpha_1 [x_1 x_3] = 0.$$

To find the last two equations, join each term in equation 4.11 with x_2 and then x_4 to obtain

$$\alpha_3 \gamma_{21} [x_3 x_2] + \alpha_4 \gamma_{22} [x_4 x_2] = 0$$

and

$$\alpha_3 \gamma_{21} [x_3 x_4] + \alpha_2 [x_2 x_4] = 0.$$

We know that this homogeneous system of linear equations in the variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ has a solution if and only if the determinant of the matrix

$$N = \begin{pmatrix} 0 & 0 & \gamma_{11}[x_3x_1] & \gamma_{12}[x_4x_1] \\ 0 & 0 & \gamma_{21}[x_3x_2] & \gamma_{22}[x_4x_2] \\ [x_1x_3] & 0 & 0 & \gamma_{12}[x_4x_3] \\ 0 & [x_2x_4] & \gamma_{21}[x_3x_4] & 0 \end{pmatrix}$$

is zero. Since,

$$|N| = [x_1x_3][x_2x_4](\gamma_{11}[x_3x_1]\gamma_{22}[x_4x_2] - \gamma_{12}[x_4x_1]\gamma_{21}[x_3x_2]),$$

we know $|N| = 0$ if and only if

$$[x_1x_3][x_2x_4](\gamma_{11}[x_3x_1]\gamma_{22}[x_4x_2] - \gamma_{12}[x_4x_1]\gamma_{21}[x_3x_2]) = 0.$$

This is obviously zero if either $[x_1x_3] = 0$ or $[x_2x_4] = 0$. Note that we could have set up this system of equatons so that the x's played the role of the y's to obtain a similar statement if either $[y_1y_3] = 0$ or $[y_2y_4] = 0$.

Otherwise, if all of $[x_1x_3], [x_2x_4], [y_1y_3]$ and $[y_2, y_4]$ are not zero we conclude $|N| = 0$ if and only if

$$\frac{\gamma_{11}\gamma_{22}}{\gamma_{12}\gamma_{21}} = \frac{[x_4x_1][x_3x_2]}{[x_3x_1][x_4x_2]}.$$

Finally, using equations 4.8 and 4.9, we arrive at the desired result. ■

One example where we can apply this theorem is in the case where the framework K_{55} realized in 3-space so that vertex set A is coplanar with no three collinear and the vertex set B is coplanar with no three collinear. We will examine the framework G obtained by removing the bars $\{a_1, b_1\}$,$\{a_1, b_2\}$,$\{a_2, b_1\}$, and $\{a_2, b_2\}$ from this realization of K_{55}. Note, by construction we have $N(A) = N(B) = 2$. Now, let P be the matrix whose columns form a basis for the space $\mathbb{D}(A)$ and whose rows are indexed by the points of A such that the first two rows are indexed by a_1 and a_2 respectively. Similarly, let Q be a matrix whose columns form a basis for the space $\mathbb{D}(B)$ and whose rows are indexed by the points of B such that the first two rows are indexed by b_1 and b_2 respectively. Denote by x_1, the point whose coordinates are the entries of the

first row of P. Similarly, denote by x_2, the point whose coordinates are the second row of P. Finally, denote by y_1 and y_2, the points which have as their coordinates the first and second rows of Q respectively. Note that since A has no three points collinear we can choose P so that x_1 and x_2 are distinct. Likewise, Q can be chosen so that y_1 and y_2 are distinct. We know that $dim(ker(\tau)_G) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B) - 4)$ if and only if the set $T = \{x_1 \otimes y_1, x_2 \otimes y_2, x_1 \otimes y_2, x_2 \otimes y_1\}$ is independent. In this case $X = \{x_1, x_2, x_1, x_2\}$ and $Y = \{y_1, y_2, y_2, y_1\}$. Now, from Theorem 4.1.7, with the above order, we may conclude T is dependent because $[x_1 x_1] = 0$. Hence removing these four bars does not decrease the dimension of the kernel by four. On the other hand, using Theorem 4.1.4 we see any subset of cardinality three of T is independent. Hence we conclude $dim(ker(\tau)_G) = dim(\mathbb{D}(A) \otimes \mathbb{D}(B) - 3) = 1$.

4.2 The Image

The most difficult part of the problem thus far, has been trying to understand the following question: If we remove one bar from a bipartite framework, when does this decrease the dimension of the image of τ by one?

We have already proved a few theorems about the image of τ, such as Theorem 3.3.9 and Theorem 3.3.11. The following are some slightly more general results about the image of τ.

THEOREM 4.2.1. *Let G_2 be a bipartite framework which is obtained from the bipartite framework G_1 by removing the bar $\{a, b\}$ from G_1. If*

$$dim(Im(\tau)_{G_2}) = dim(Im(\tau)_{G_1}) - 1$$

then every element w of the $ker(\tau)_{G_1}$ has $w_{ab} = 0$.

Proof. We prove the contrapositive. Assume there is an element w of the kernel of τ with $w_{ab} = 1$. Then there is a scalar μ_λ for each element $\lambda \in \Omega_{G_1}$ such that the ab position of $\lambda - \mu_\lambda w$ is zero. Obviously,

$$\Omega_{G_2} = \{\lambda - \mu_\lambda w | \lambda \in \Omega_{G_1}\}.$$

Furthermore, adding an element from the kernel of τ to any element of Ω_{G_1} does not change its image under τ. Hence we can show $dim(Im(\tau)_{G_2}) = dim(Im(\tau)_{G_1})$. To see this, note that

$$Im(\tau)_{G_2} \subset Im(\tau)_{G_1}.$$

Now, let $\eta \in Im(\tau)_{G_1}$, then there is $\lambda \in \Omega_{G_1}$ such that $\tau(\lambda) = \eta$. Furthermore, $\lambda - \mu_\lambda w \in \Omega_{G_2}$ and

$$\tau(\lambda - \mu_\lambda w) = \tau(\lambda) = \eta.$$

Hence we may conclude $dim(Im(\tau)_{G_2}) = dim(Im(\tau)_{G_1})$. ■

THEOREM 4.2.2. *Let G be a framework which is obtained by removing the set*

$$\mathcal{B} = \{\{a_{i_1}, b_{j_1}\}, \{a_{i_2}, b_{j_2}\}, \{a_{i_3}, b_{j_3}\}, ..., \{a_{i_l}, b_{j_l}\}\}$$

of l distinct bars from K_{mn}. Then $dim(Im(\tau)_G) = dim(\mathbb{D}(C_{K_{mn}} \otimes C_{K_{mn}}) - l$ if and only if the following two conditions are satisfied:

1. Every element of $ker(\tau)_{K_{mn}}$ is zero in the positions associated with the bars in $\mathcal{B}$.

2. There exist an independent set $\mathcal{E} = \{e_1, e_2, e_3, ..., e_l, e_{l+1}, ..., e_p\}$ such that $\mathcal{E}$ together with a basis for $ker(\tau)_{K_{mn}}$ form a basis for $\Omega_{K_{mn}}$ and for each $k = 1, 2, 3, ..., l$ there is a one in the $a_{i_k} b_{j_k}$ position of e_k and a zero in the positions of e_k associated with the bars of $\mathcal{B} - \{\{a_{i_k}, b_{j_k}\}\}$.

The proof of the above theorem follows from Theorem 4.2.1 and the techniques of Theorem 4.1.2.

Unfortunately, it may be difficult to tell, by observation, whether or not a realization of a particular framework satisfies either of the previous theorems.

CHAPTER 5
EXAMPLES

5.1 K_{44} Minus One Bar

Let G be a framework obtained by removing the bar $\{a_1, b_1\}$ from the bipartite framework K_{44} realized in the plane. Using the results in this paper, we are able to determine the dimension of the stress space for any realization of the above framework.

The following is a complete list of possible dimensions of the stress space of G together, with a breakdown of the image and kernel dimensions and the realizations which yield these dimensions.

We will label each realization according to the following cases:

Case I: In this case we will assume no three points of either the set A or the set B are collinear.

Case II: Assume the set A has three points collinear, yet A still spans the plane and the set B has no three points collinear.

Case III: Assume both sets still span the plane and both set have three points collinear.

Case IV: Assume A is collinear and B has no three collinear.

Case V: Assume A is collinear and B has only three points collinear.

Case VI: Assume both sets are collinear.

For simplicity the results of each case are listed in the tables to follow. Furthermore, to aid the reader, following the two tables we have included a list of figures corresponding to each realization. In each case, we have tried to include a figure which depicts the most general realization satisfying the given conditions; yet within a case there may be special realizations which yield the same dimension stress space.

Table 5.1: Complete Description of G

Realization	$dim(\Omega)$	N(A)N(B)	$\lvert C_G \rvert$	$ker(\tau)$	$Im(\tau)$
I.1 Generic Position	2	1	8	0	2
I.2 All eight points on a conic	3	1	8	0	3
II.1 A has three points collinear, a_1 an isthmus of A, C_G not on a conic, $[b_1a_3a_4] \neq 0$	2	1	7	1	1
II.2 As above with all of C_G on a conic (necessarily degenerate).	3	1	7	1	2
II.3 As in II.1 with $[b_1a_3a_4] = 0$ and C_G not on a conic	3	1	8	1	2
II.4 As in II.3 except all of C_G on a conic.	4	1	8	1	3
II.5 A has three points collinear, a_1 not an isthmus of A, and not all of C_G on a conic.	2	1	8	0	2
II.6 As in II.5 except all of C_G fall on a conic.	3	1	8	0	3
III.1 Both sets A and B have only three points collinear, but on different lines, both a_1 and b_1 isthmuses, $[a_1b_3b_4] \neq 0$ and $[b_1a_3a_4] \neq 0$	2	1	6	1	1
III.2 As in III.1 with the 3 collinear points of A on the same line as the 3 collinear points of B.	4	1	6	1	3
III.3 As in III.1 with $[a_1b_3b_4] = 0$ or $[b_1a_3a_4] = 0$.	3	1	7	1	2
III.4 As in III.1 with both $[a_1b_3b_4] = 0$ and $[b_1a_3a_4] = 0$. (note here all eight points are on a degenerate conic)	4	1	8	1	3
III.5 A and B both have three points collinear, but on different lines, both sets also span the plane, only one of a_1 or b_1 an isthmus (say a_1 is an isthmus of A and b_2 an isthmus of B), $[b_1a_3a_4] \neq 0$.	2	1	7	1	1
III.6 As in III.5 but all of C_G on conic, that is $[b_2a_2a_3] = 0$.	3	1	7	1	2

Table 5.1–continued

Realization	$dim(\Omega)$	N(A)N(B)	$\lvert C_G \rvert$	$ker(\tau)$	$Im(\tau)$
III.7 As in III.5 with the three collinear points of A on the same line as the three collinear points of B.	4	1	7	1	3
III.8 As in III.5 with $[b_1 a_3 a_4] = 0$, not all of C_G on a conic.	3	1	8	1	2
III.9 As in III.8 with all of C_G on a conic.	4	1	8	1	3
III.10 Again, both A and B have three points collinear, neither a_1 nor b_1 an isthmus, not all of C_G on a conic.	2	1	8	0	2
III.11 As in III.10 with all of C_G on a conic.	3	1	8	0	3
IV.1 A is collinear, B has no three collinear, $B \cap < A >= \emptyset$	2	2	4	1	1
IV.2 As in IV.1 together with $\lvert B \cap < A > \rvert = 1$.	3	2	5	1	2
IV.3 As in IV.1 together with $\lvert B \cap < A > \rvert = 2$.	4	2	6	1	3
V.1 A collinear, B has three points collinear, with b_1 an isthmus of B, $B \cap < A >= \emptyset$, and $[a_1 b_3 b_4] \neq 0$.	2	2	3	2	0
V.2 As in V.1 with $[a_1 b_3 b_4] = 0$.	3	2	4	2	1
V.3 As in V.1 together with $\lvert B \cap < A > \rvert = 1$.	3	2	4	2	1
V.4 As in V.1 together with $\lvert B \cap < A > \rvert = 1$, and $[a_1 b_3 b_4] = 0$.	4	2	5	2	2
V.5 As in V.1 together with $\lvert B \cap < A > \rvert = 3$.	6	2	7	2	4
VI.1 Both A and B collinear with $< A > \cap B = \emptyset$ and $A \cap < B >= \emptyset$.	3	4	0	3	0
VI.2 As in VI.1 with the two lines intersecting in either one point of A or one point of B.	3	4	1	3	0
VI.3 $A \cup B$ collinear.	8	4	8	3	5

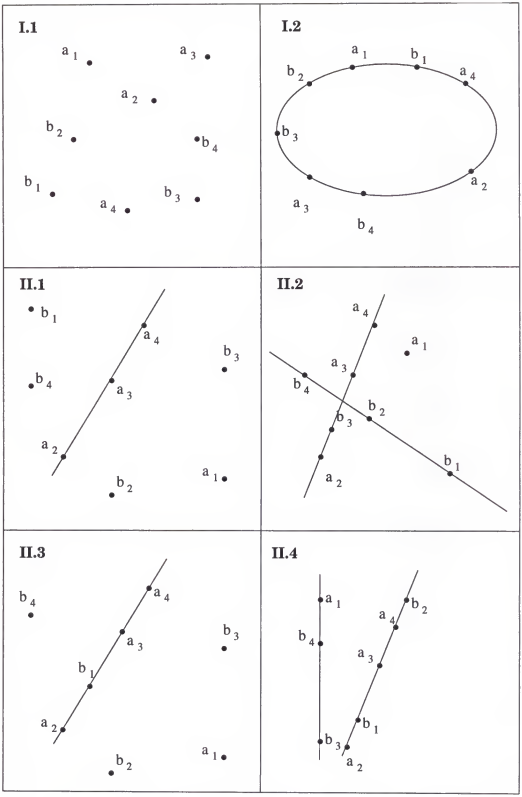


Figure 5.1: Realizations of K_{44}

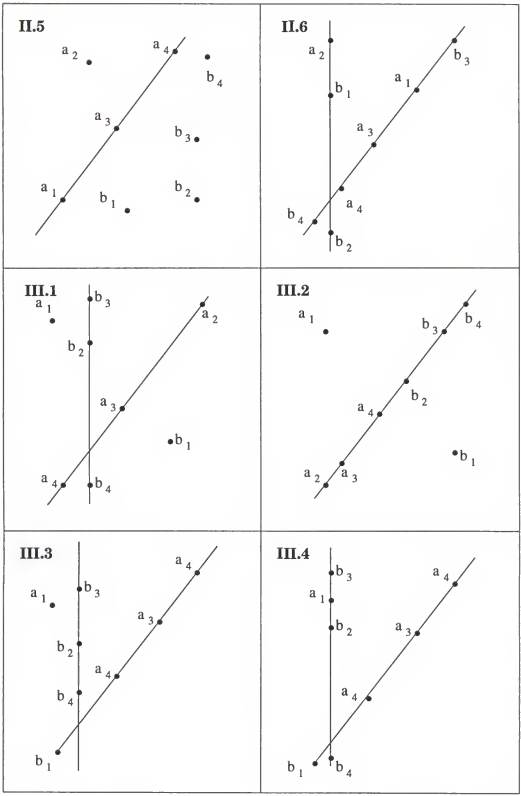


Figure 5.1–continued

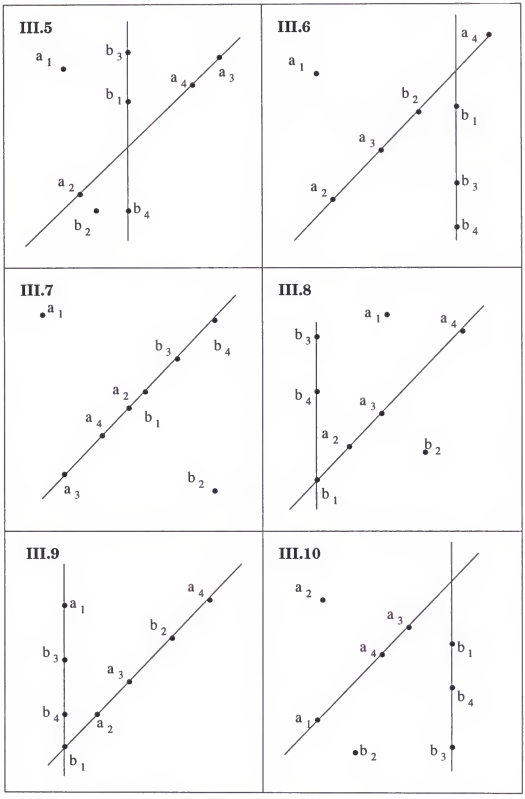


Figure 5.1–continued

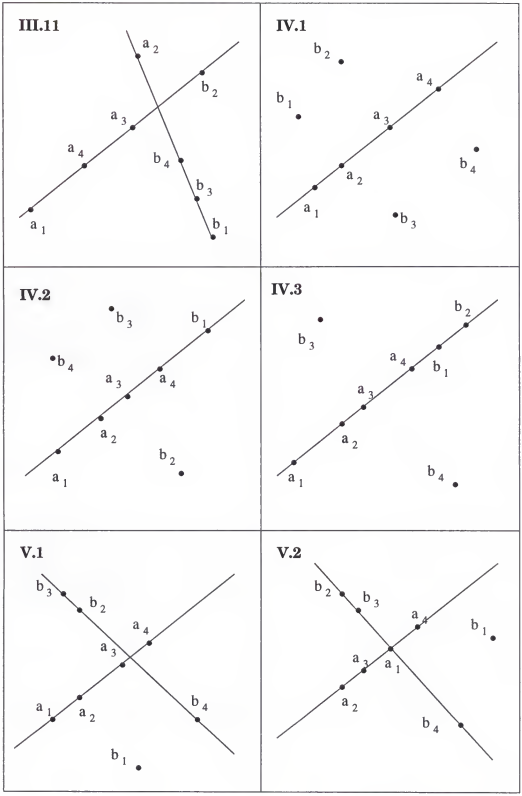


Figure 5.1–continued

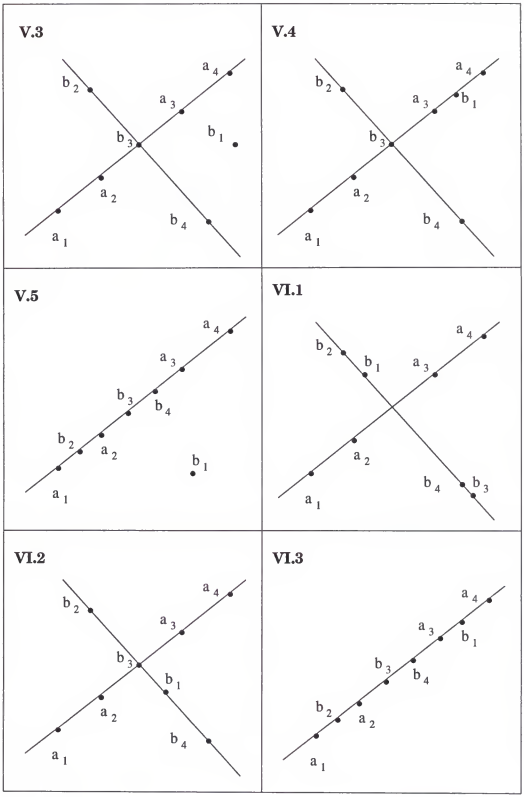


Figure 5.1–continued

5.2 K_{44} Minus Two Bars

The following tables are similar to the tables presented previously except they are a complete listing the dimension of the stress space of the framework G_2 obtained by removing the two bars $\{a_1, b_1\}$ and $\{a_2, b_2\}$ from the complete bipartite framework K_{44} realized in the plane. We have already noted that there are realizations of this framework for which no theorem applies. Hence, in these cases, the entries in the tables were either worked out by hand or by computer using the original definition of the rigidity matrix. Furthermore, in the tables to follow two columns have been added. The first extra column indicates the dimension of $\mathbb{D}(C_G \otimes C_G)$, denoted simply by $\mathbb{D}$ in the table. This is added since in some realizations we have $Im(\tau) \neq \mathbb{D}(C_G \otimes C_G)$. We have also added a column which indicates if any theorem applies. Also, as before, following the tables we have included figures which correspond to each of the listed realizations.

Table 5.2: Complete Description of G_2

Realization	Ω	$N_A N_B$	$\lvert C_G \rvert$	K_τ	I_τ	$\mathbb{D}$	Theorem
I.1 Both sets A and B are in generic position in the plane.	1	1	8	0	1	2	n
I.2 The points $a_1, a_3, a_4, b_2, b_3, b_4$ on a conic and the points $a_2, a_3, a_4, b_1, b_3, b_4$ on an independent conic.	2	1	8	0	2	2	y
I.3 Both sets A and B have no three points collinear. All eight points of C_G on the same conic.	2	1	8	0	2	3	n
II.1 The set A has three points collinear, but still spans the plane. B has no three collinear. a_1 or a_2 the isthmus of A.	1	1	7	0	1	1	y
II.2 As in II.1 with all of C_G on a conic.	2	1	7	0	2	2	y
II.3 As in II.1 with $[b_1 a_3 a_4] = 0$.	2	1	8	0	2	2	y
II.4 As in II.3 with all of C_G on a conic.	3	1	8	0	3	3	y
II.5 A has three collinear, but still spans the plane. B has no three collinear and a_3 or a_4 the isthmus of A.	1	1	8	0	1	2	n
II.6 As in II.5 with all of C_G on a conic.	2	1	8	0	2	3	n
III.1 A has three points collinear and B has three points collinear and both sets still span the plane. a_1 the isthmus of A and b_2 the isthmus of B (similar if a_2 and b_1 are the isthmuses).	1	1	6	1	0	0	y
III.2 As in III.1 with $[b_1, a_2, a_3] = 0$ (or $[a_2 b_3 b_4] = 0$).	2	1	7	1	1	1	y
III.3 As in III.1 with all of C_G on a conic.	2	1	6	1	1	1	y
III.4 As in III.3 with $[b_1, a_2, a_3] = 0$ (or $[a_2 b_3 b_4] = 0$).	3	1	7	1	2	2	y

Table 5.2–continued

Complete Description of G_2							
Realization	Ω	$N_A N_B$	$\lvert C_G \rvert$	K_τ	I_τ	$\mathbb{D}$	**Theorem**
III.5 As in III.1 with the three collinear points of A on the same line formed by the three collinear points of B.	4	1	8	1	3	3	y
III.6 A and B three points collinear each and both span the plane. a_1 and b_1 are the isthmuses. (similar if a_2 and b_2 are the isthmuses).	1	1	6	0	1	1	y
III.7 As in III.6 with $[a_1 b_3 b_4] = 0$ (or $[b_1 a_3 a_4] = 0$).	2	1	7	0	2	2	y
III.8 As in III.6 with $[a_1 b_3 b_4] = 0$ and $[b_1 a_3 a_4] = 0$.	3	1	8	0	3	3	y
III.9 As in III.6 with the three points collinear in A on the same line spanned by the three collinear points of B.	3	1	6	0	3	3	y
III.10 As in III.6 with a_1 and b_3 the isthmuses. (any combination of a_i and b_j (or b_i and a_j) with $i = 1, 2$ and $j = 3, 4$ will yield a similar result). Same result if exactly one of $[a_1 b_2 b_4] = 0$ or $[b_3 a_2 a_3] = 0$.	1	1	7	0	1	1	y
III.11 As in III.10 with $[a_1 b_2 b_4] = 0$ and $[b_3 a_2 a_3] = 0$.	2	1	7	0	2	2	y
III.12 As in III.10 with $[b_1 a_2 a_3] = 0$ and exactly one of $[a_1 b_2 b_4] = 0$ or $[b_3 a_2 a_3] = 0$.	2	1	8	0	2	2	y
III.13 As in III.10 with $[b_1 a_2 a_3] = 0$, $[a_1 b_2 b_4] = 0$ and $[b_3 a_2 a_3] = 0$.	3	1	8	0	3	3	y
III.14 As in III.6 with none of a_1, a_2, b_1, b_2 isthmuses. Say a_4 and b_4 the isthmuses.	1	1	8	0	1	2	n
III.15 As in III.14 with both $[a_4 b_1 b_2] = 0$ and $[b_4 a_1 a_2] = 0$, hence all of C_G on a conic.	2	1	8	0	2	3	n
IV.1 A collinear, B has no three points collinear.	1	2	4	0	1	1	y

Table 5.2–continued

Complete Description of G_2							
Realization	Ω	$N_A N_B$	$\lvert C_G \rvert$	K_τ	I_τ	$\mathbb{D}$	**Theorem**
IV.2 As in IV.1 with any one point of B on the line spanned by the set A.	2	2	5	0	2	2	y
IV.3 As in IV.1 with any two points of B on the line spanned by the set A.	3	2	6	0	3	3	y
IV.4 A is collinear and B has three points collinear. b_1 (or b_2) an isthmus of B.	1	2	3	1	0	0	y
IV.5 As in IV.4 with any one point of B on the line spanned by A.	2	2	4	1	1	1	y
IV.6 As in IV.4 with $[b_1 a_1 a_2] = 0$ and any other point of B on the line spanned by A.	3	2	5	1	2	2	y
IV.7 As in IV.4 with $[a_1 b_2 b_3] = 0$.	2	2	4	1	1	1	y
IV.8 As in IV.7 with b_1 on the line spanned by the set A.	3	2	5	1	2	2	y
IV.9 As in IV.4 with the three collinear points of B spanning the same line as the set A.	5	2	7	1	4	4	y
IV.10 As in IV.4 except with b_3 (or b_4) the isthmus of B instead of b_1.	1	2	4	0	1	1	y
IV.11 As in IV.10 with any point of B on the line spanned by the set A.	2	2	5	0	2	2	y
IV.12 As in IV.10 with b_3 and any other point of B on the line spanned by the set A.	3	2	6	0	3	3	y
IV.13 As in IV.10 with the three collinear points of B falling on the line spanned by the set A.	4	2	7	0	4	4	y
V.1 The set A is collinear and the set B is collinear.	2	4	0	2	0	0	y
V.2 As in V.1 with one point of A on the line spanned by the set B, or one point of B on the line spanned by the set A.	2	4	1	2	0	0	y
V.3 As in V.1 with the line spanned by the set A equal to the line spanned by the set B.	7	4	8	2	5	5	y

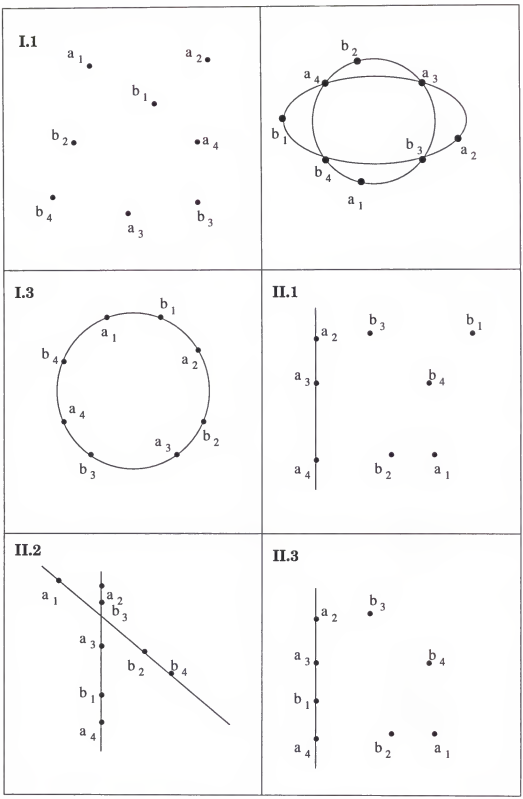


Figure 5.2: Realizations of K_{44}

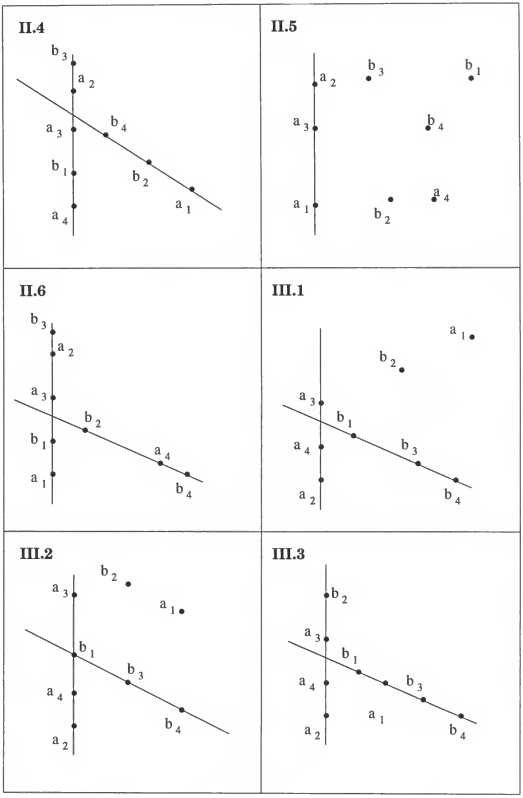
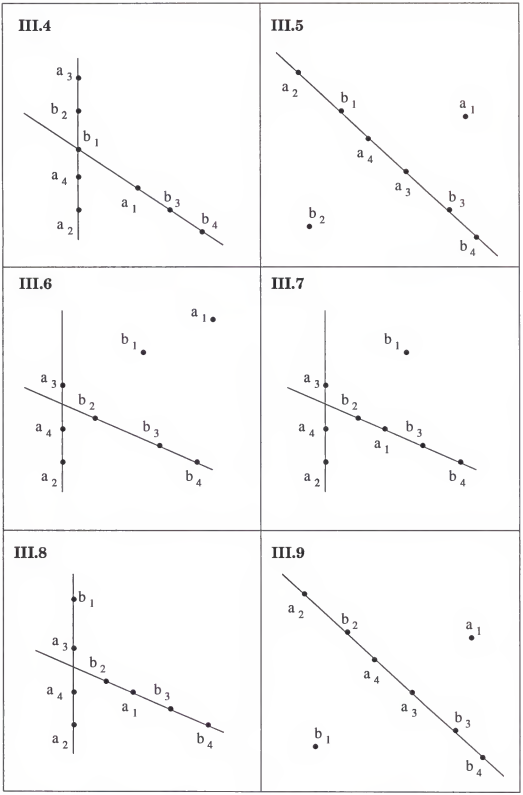


Figure 5.2–continued



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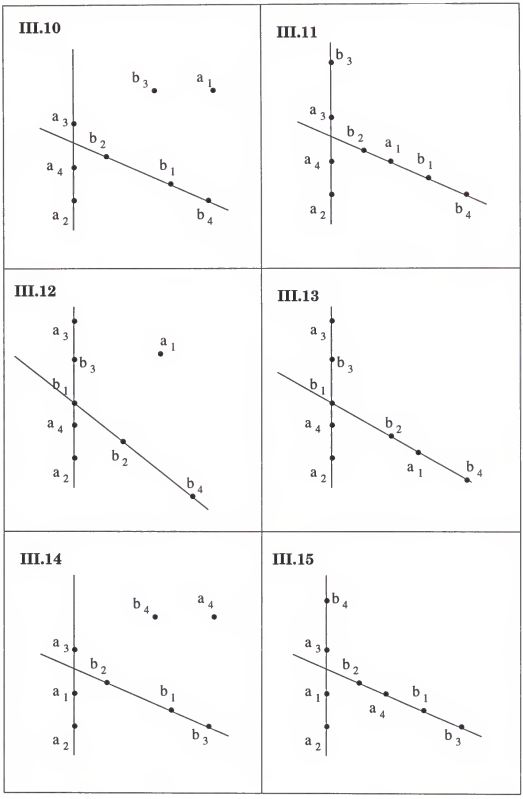

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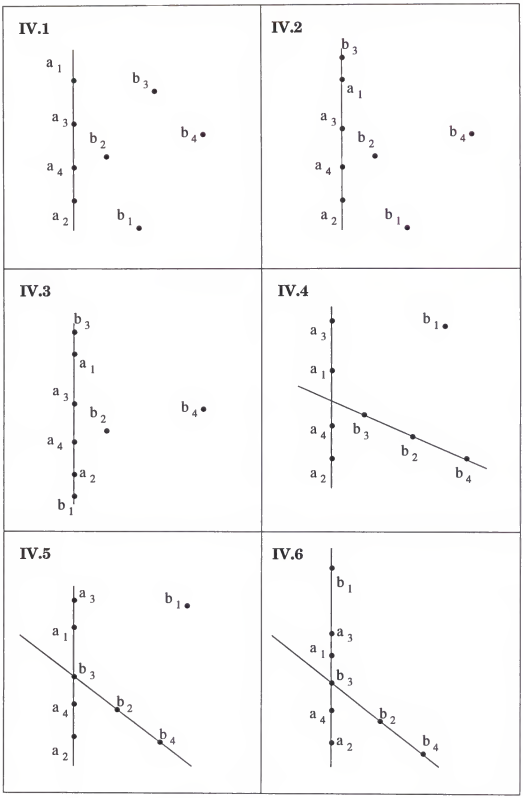

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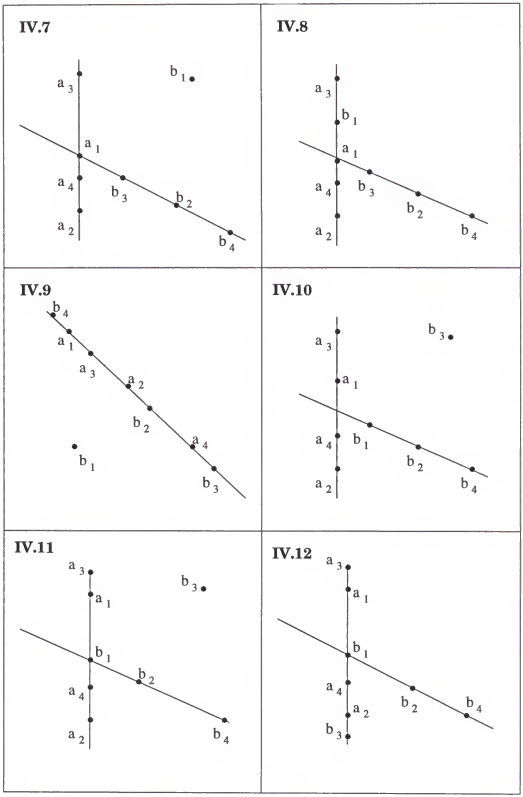

Figure 5.2–continued

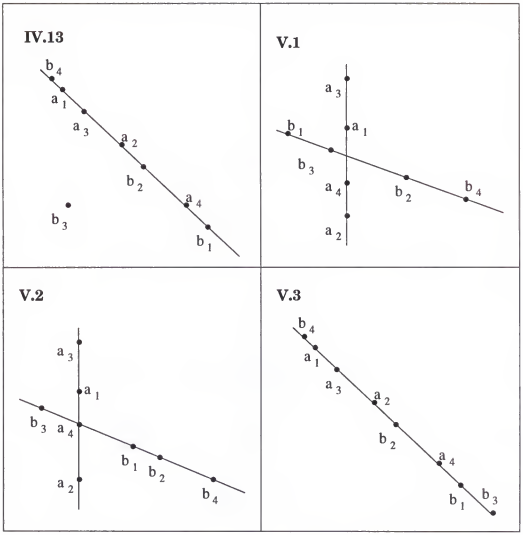

Figure 5.2–continued

CHAPTER 6
CONCLUSION

6.1 Summary

In this dissertation we wanted to extend the result of Bolker and Roth to general bipartite frameworks. To this end we stated and proved theorems for classes of bipartite frameworks which are not complete. In particular, Corollary 3.2.4 gives the dimension of the stress space for any realization of a framework which has a complete bipartite spanning subframework. Examples of frameworks where this corollary applies were given. We also gave two realizations of the same framework which had no complete bipartite spanning subframeworks. The point of these examples was to show that the general problem may be very difficult. These examples also show that there may be some complicated geometric condition on the joints that, if satisfied, will change the dimension of the stress space.

There is one class of bipartite frameworks for which we achieved what we set out to do. Corollary 3.3.5 gives the dimension of the stress space for any realization of a bipartite framework which is obtained by removing l bars, all incident at one joint, from the complete bipartite framework K_{mn}.

We note here, that these theorems do more than simply calculate the dimension of the stress space (this can be calculated using the rigidity matrix). These theorems also enable us to predict which geometric conditions will yield special realizations for which the framework at hand would not be rigid. For instance we can easily see that the framework obtained by removing a bar from the framework K_{34} will be rigid unless the K_{33} subframework is not rigid.

In Chapter 4 we presented what is known about general bipartite frameworks. We stated theorems about $dim(Im(\tau)_G)$ and $dim(ker(\tau)_G)$. The latter includes a short sidetrack into tensor products.

Finally in Chapter 5 we presented, in tabular form, two examples worked out in full detail. The point of these examples is twofold. First they show the usefulness of our results in determining the dimension of the stress space of some non-complete bipartite frameworks and how changing the realization can change this dimension. Second they show how complicated the realizations of a framework can become even in a relatively simple case, realized in the plane.

6.2 Future Research

At the end of chapter 4 we note that not much is known about $dim(Im(\tau)_G)$ outside the cases stated earlier in this dissertation. In fact, the question of dimension is not suitably answered for $ker(\tau)_G$ in the general case either. Our hope is to continue to make progress on these problems and answer these questions in full. Some techniques which seem hopeful involve the theory of matroids [14, 15]. We have been successful in describing the kernel of τ as a matroid, but again the image proves to be difficult.

The problems solved in this dissertation are an obvious extension of the work of Bolker and Roth. This is not the only generalization. In fact, using a higher order generalization of rigidity, called $r - rigidity$, presented by Tay et al. [11, 12], one can hope for a results similar to those of Bolker and Roth for complete tripartite frameworks. We have done some work in this area and it does not seem to be a trivial generalization.

Let A, B and C all be finite subsets of $\mathbb{P}^d$ with cardinalities k, m, n respectively. Denote by $\Omega(A, B, C)$ the space of 3-stresses of the complete tripartite framework with underlying graph K_{kmn}. Define, as in the bipartite case, a map $\tau : \Omega(A, B, C) \to \mathbb{R}^t$.

PROPOSITION 6.2.1. *Given a tripartite framework with vertex sets A, B and C,*

$$dim(\Omega(A,B,C)) = dim(ker(\tau)) + dim(Im(\tau)).$$

We have already shown the following:

PROPOSITION 6.2.2. *Given a tripartite framework with vertex sets A, B and C then*

$$\mathbb{D}(A) \otimes \mathbb{D}(B) \otimes \mathbb{D}(C) \subset \Omega(A,B,C).$$

PROPOSITION 6.2.3. *Given a tripartite framework with vertex sets A, B and C*

1. $\mathbb{D}(A) \otimes \Omega(B,C) \subset \Omega(A,B,C)$
2. $\mathbb{D}(B) \otimes \Omega(A,C) \subset \Omega(A,B,C)$
3. $\mathbb{D}(C) \otimes \Omega(A,B) \subset \Omega(A,B,C)$

We have also shown that, in general, if $E \subset A \cup B \cup C$, then $Im(\tau)$ is not a subset $\mathbb{D}(E \otimes E \otimes E)$ for any of the obvious choices of τ. We would like to find a linear map τ for which we get results analogous to the results of Bolker and Roth.

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BIOGRAPHICAL SKETCH

Kevin Peterson was born in Springfield, Ohio, on May 16, 1961. He received an Associate of Arts degree from Broward Community College. He spent a year at Florida Atlantic University, transferred to the University of Central Florida and received a Bachelor of Science degree in mathematics. During the summer of 1987 he married Vicky Crowdis. Kevin attended the University of Florida where he received a Master of Science and a Ph. D. degrees in mathematics under the direction of Dr. Neil White.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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